

FILLING LINKS AND ESSENTIAL SYSTOLE

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ABSTRACT. We answer a question of Freedman and Krushkal, producing *filling links* in any closed, orientable 3-manifold. The links we construct are hyperbolic, and have large *essential systole*, contrasting earlier geometric constraints on hyperbolic links in 3-manifolds due to Adams and Reid and Lakeland and the first author.

1. INTRODUCTION

Given a closed 3-manifold, M , a 1-spine for M (also called a *carrier graph*) is a graph Γ with $\text{rank}(\pi_1(\Gamma)) = \text{rank}(\pi_1(M))$ and map $f: \Gamma \rightarrow M$ such that $f_*: \pi_1(\Gamma) \rightarrow \pi_1(M)$ is surjective. Given a link $L \subset M$, let $i: M \setminus L \rightarrow M$ be the inclusion. We say that $f: \Gamma \rightarrow M \setminus L$ is an L -relative 1-spine if $i \circ f: \Gamma \rightarrow M$ is a 1-spine. We say that L is a *filling link* if for every L -relative 1-spine $f: \Gamma \rightarrow M \setminus L$, we have

$$f_*: \pi_1(\Gamma) \rightarrow \pi_1(M \setminus L)$$

is injective. This paper is motivated by the following.

Question 1.1 (Freedman-Krushkal [FK23]). *Does every closed 3-manifold contain a filling link?*

In the appendix to [FK23], the first author and Reid answered this question affirmatively for a closed, orientable 3-manifold M for which the rank of $\pi_1(M)$ is 2. Following this, Stagner proved that the answer is also yes when $\pi_1(M)$ has rank 3 [Sta21]. In this paper, we answer the question affirmatively for an arbitrary closed, orientable 3-manifold.

Theorem 1.2. *Every closed, orientable 3-manifold M contains a filling link.*

The filling links $L \subset M$ we construct to prove Theorem 1.2 are *hyperbolic*; that is, $M \setminus L$ admits a complete hyperbolic structure (which is unique by Mostow-Prasad Rigidity [Mos73, Pra73]). The idea is to find a hyperbolic link L so that for any L -relative 1-spine $f: \Gamma \rightarrow M \setminus L$ and any basis for $\pi_1(\Gamma)$, the f_* -image of the basis elements have large translation length; hyperbolic geometry then forces f_* to be injective.

The intuition for this construction comes from the work of White [Whi02] and independently Kapovich and Weidmann [KW03] (see below). Recall that the *systole* of a hyperbolic 3-manifold is the length of the shortest closed geodesic. White proved that the systole of a closed hyperbolic 3-manifold M is bounded by a function of the rank of its fundamental group. His proof involves analyzing 1-spines $f: \Gamma \rightarrow M$ of minimal length, and then showing that if the systole is sufficiently

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large, then f_* must be injective, contradicting the fact it is a 1-spine of a closed hyperbolic 3-manifold. For more on probing the geometry of hyperbolic 3-manifolds via their 1-spines, see [BCW04, Bir09, BS11], for example.

The first guess for constructing the required links to prove Theorem 1.2 might thus be to find hyperbolic links $L \subset M$ such that the systole of $M \setminus L$ is sufficiently large. This runs into a theorem of Adams and Reid [AR00] which states that **any** hyperbolic link in a closed, non-hyperbolic 3-manifold has systole bounded above by 7.35534.... This precise strategy even fails for closed hyperbolic 3-manifolds by a theorem of Lakeland and the first author [LL14], which bounds the length of the systole of hyperbolic link complement by a function of the volume of the original closed hyperbolic manifold.

The short closed geodesics which are used to illustrate the uniform upper bound on systoles in both [AR00] and [LL14] are typically null-homotopic in the 3-manifold. This hints at a more refined notion of systoles for hyperbolic links. Specifically, if $L \subset M$ is a hyperbolic link, we define the *essential systole* of L to be

$$\text{esssys}(L) = \inf\{\ell(\gamma) \mid \gamma \text{ is a loop in } M \setminus L \text{ with } i \circ \gamma \text{ non-null-homotopic in } M\}$$

Remark 1.3. *As a consequence of our definition, observe that if $\text{esssys}(L) > 0$, then every parabolic element of $\pi_1(M \setminus L)$ is necessarily in the kernel of the induced map $i_*: \pi_1(M \setminus L) \rightarrow \pi_1(M)$ from inclusion, since such an element is represented by loops with arbitrarily small length.*

White's theorem can be seen as a consequence of a theorem of Kapovich and Weidmann [KW03], which appeared at roughly the same time as White's paper [Whi02]. We appeal to this result of Kapovich and Weidmann (stated as Theorem 2.2 below), and easily deduce the following.

Theorem 1.4. *Given $n > 0$, there exists $R > 0$ so that if M is a closed 3 manifold with $\text{rank}(\pi_1(M)) = n$, and $L \subset M$ is a hyperbolic link with $\text{esssys}(L) > R$, then L is a filling link.*

Theorem 1.4 thus reduces the problem of finding filling links to the problem of finding hyperbolic links with large essential systole. We do this by an explicit construction, proving the following.

Theorem 1.5. *Given a closed, orientable 3-manifold M with $\text{rank}(\pi_1(M)) \geq 1$ and $r > 0$, there exists a hyperbolic link $L \subset M$ such that*

$$\text{esssys}(L) > r.$$

Theorem 1.2 is now an easy consequence of these two theorems.

Proof of Theorem 1.2 assuming Theorems 1.4 and 1.5. Suppose $\text{rank}(\pi_1(M)) = n$ and let $R > 0$ be as in Theorem 1.4. By Theorem 1.5, there exists a link $L \subset M$ so that $\text{esssys}(L) > R$, which is thus filling by Theorem 1.4. \square

Outline of the paper and sketch of the proof. In Section 2 we recall some basics about Nielsen equivalence, §2.1, and hyperbolic geometry, §2.2, then conclude by proving Theorem 1.4 in §2.3. The remainder of the paper, Section 3 contains the construction of links with large essential systole. The first step is to construct a triangulation of our arbitrary closed manifold M with controlled local properties, but large *combinatorial systole*, following a construction due to Cooper and

Thurston [CT88]; see §3.1. Next, in §3.2, we explicitly construct a tangle in a tetrahedron whose complement admits a nice hyperbolic structure with totally geodesic boundary and dihedral angles equal to $\frac{\pi}{2}$. This tangle serves as the building block for the link with large essential systole that we construct in §3.3. Specifically, starting with the triangulation τ of a closed 3-manifold M constructed in §3.1 having large combinatorial systole, we delete the tangle from each tetrahedron. The resulting link L_τ has an explicit, complete $\text{CAT}(-1)$ metric, and hence admits a complete hyperbolic metric by Thurston's Hyperbolization Theorem [Thu86]. While we do not have explicit control over the hyperbolic metric, we use the $\text{CAT}(-1)$ metric to analyze a family of surfaces with bounded Euler characteristic which serve as "barriers" to accessing the tetrahedra. We show that having large combinatorial systole implies any loop γ in $M \setminus L_\tau$ which is non-null-homotopic in M must intersect many tetrahedra, and hence many of the surfaces. Every intersection of γ with one of the surfaces passes through a point with bounded injectivity radius, and separation properties of the surfaces imply that γ must pass through many such distinct points, and is therefore long.

Remark 1.6. *Because Kapovich-Weidmann work in the setting of arbitrary δ -hyperbolic spaces, we could avoid using Thurston's Hyperbolization Theorem to prove that the links we construct are filling, and shorten the argument a little bit; see Section 4. Theorem 1.5 seems independently interesting due to its contrast with the results of [AR00] and [LL14], so we kept the slightly longer proof.*

Acknowledgements. The authors would like to thank Alan Reid for helpful conversations, including the reference to [Sar18] mentioned at the end of the paper. The first author is particularly grateful for his collaboration with Reid on the appendix of [FK23] and the discussions during that time regarding filling links that have influenced this paper.

2. HYPERBOLIC GEOMETRY AND LARGE ESSENTIAL SYSTOLE

The goal of this section is to prove the following.

Theorem 1.4 *Given $n > 0$, there exists $R > 0$ so that if M is a closed 3 manifold with $\text{rank}(\pi_1(M)) = n$, and $L \subset M$ is a hyperbolic link with $\text{esssys}(L) > R$, then L is a filling link.*

2.1. Nielsen equivalence. Nielsen transformations were originally introduced by Nielsen in [Nie24] to prove that every subgroup of a finitely generated free group is free. Since then, they have been used extensively to study properties of free groups (see [FRS95]). In this section, we recall basic notions from [Nie24].

Definition 1 (Nielsen Equivalence). *Let G be a group, and consider an n -tuple of elements, $M = (g_1, g_2, g_3, \dots, g_n) \in G^n$. Define the following elementary Nielsen moves on M :*

- (1) *For some $1 \leq i \leq n$, replace g_i with g_i^{-1} in M ;*
- (2) *For $i \neq j$, $1 \leq i, j \leq n$, replace g_i with $g_i g_j$ in M ;*
- (3) *For $i \neq j$, $1 \leq i, j \leq n$, interchange g_i and g_j in M .*

A Nielsen transformation is a finite sequence of elementary Nielsen moves. We say that $M, M' \in G^n$ are Nielsen equivalent, denoted $M \sim_N M'$, if they differ by a Nielsen transformation.

Recall that the *basis* of a finite rank free group is a minimal collection of generators of \mathbb{F}_n . In [Nie24], Nielsen showed that if G is a finite rank free group, then the set of Nielsen transformations on a basis generates $\text{Aut}(G)$. The following lemma is immediate.

Lemma 2.1. *Let $\phi \in \text{Hom}(\mathbb{F}_n, G)$, where \mathbb{F}_n is a finite rank free group. Given a basis $\{b_1, b_2, \dots, b_n\}$ of \mathbb{F}_n , let $c_i = \phi(b_i)$ ($1 \leq i \leq n$), and let $L_1 = (c_1, c_2, \dots, c_n) \in G^n$. Let $L_2 = (c'_1, c'_2, \dots, c'_n) \in G^n$ such that $L_1 \sim_N L_2$. Then there exists $\{b'_1, b'_2, \dots, b'_n\}$, a basis for \mathbb{F}_n , such that $\phi(b'_i) = c'_i$ for all $1 \leq i \leq n$.*

Proof. Note that $c_i = \phi(b_i)$ implies that $c_i^{-1} = \phi(b_i^{-1})$ and $c_i c_j = \phi(b_i b_j)$. Thus, if $L_1 = (c_1, c_2, \dots, c_n)$ and $L_2 = (c'_1, c'_2, \dots, c'_n)$ differs from L_1 by a Nielsen transformation, then we can lift the Nielsen transformation to (b_1, \dots, b_n) to produce $(b'_1, b'_2, \dots, b'_n)$ so that $\phi(b'_i) = c'_i$ for all i . The resulting tuple $(b'_1, b'_2, \dots, b'_n)$ differs from (b_1, b_2, \dots, b_n) by a Nielsen transformation, which is an element of $\text{Aut}(\mathbb{F}_n)$. Therefore $(b'_1, b'_2, \dots, b'_n)$ is also a basis of \mathbb{F}_n . \square

2.2. Hyperbolic space. In this section, we recall basic facts about hyperbolic space which will be useful in the proof of Theorem 1.4. See, for example, Chapter 2 of [Thu97].

We consider the upper-half space model of *hyperbolic 3-space*,

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$$

endowed with the Riemannian metric $\frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}$. We let ρ denote the associated distance function. The group of orientation-preserving isometries of (\mathbb{H}^3, ρ) is $\text{PSL}_2(\mathbb{C})$, the set of 2×2 matrices with determinant 1 and complex entries, modulo $\{\pm I\}$. The action is by conformal extension of linear fractional transformations on the (x_1, x_2) -plane, viewed as \mathbb{C} . A *hyperbolic 3-manifold* is the quotient of \mathbb{H}^3 by a discrete, torsion-free subgroup of $\text{PSL}_2(\mathbb{C})$.

Recall that a geodesic metric space, X , is δ -hyperbolic if for any $x, y, z \in X$, the geodesic segment $[x, y]$ lies in the δ -neighborhood of the union of geodesic segments $[x, z] \cup [y, z]$. In other words, all triangles in X are δ -thin.

Since δ -hyperbolic spaces are modeled on hyperbolic space, \mathbb{H}^3 is a classic example of a δ -hyperbolic space. To see this, it suffices to consider a triangle in the upper half plane \mathbb{H}^2 obtained by restricting ρ to the subspace with $x_1 = 0$. Triangles in \mathbb{H}^2 have area $\pi - \alpha - \beta - \gamma$, where α, β , and γ are the interior angles of the triangle. The maximum area of a triangle in \mathbb{H}^2 is π , which is realized by an ideal triangle with vertices at $-1, 1$, and ∞ , for example. Note that δ is the distance between i and the geodesic $x_2 = 1$. This distance is bounded above by the distance between i and $1 + i$, which is $\ln\left(\frac{1}{\sqrt{2}}\right) < 1$. So we can set $\delta = 1$ to be a δ -hyperbolicity constant for \mathbb{H}^3 .

2.3. Large essential systole implies filling. We will deduce Theorem 1.4 from a general theorem of Kapovich-Weidmann from [KW03] about actions on Gromov hyperbolic spaces (whose origins they attribute to Gromov). We will only need to apply their result in the case that the Gromov hyperbolic space is \mathbb{H}^3 , and will not need the full strength of their conclusion. We therefore state only the form we will need, but emphasize that their result is more general and has a stronger conclusion.

Theorem 2.2 ([KW03]). *For any integer $n > 0$, there exist a constant $C(n)$ with the following property. Suppose a group*

$$G = \langle g_1, \dots, g_n \rangle < \mathrm{PSL}_2(\mathbb{C})$$

Then one of the following holds:

- (1) *The group G is free with basis (g_1, \dots, g_n) , or*
- (2) *The n -tuple (g_1, \dots, g_n) is Nielsen equivalent to (g'_1, \dots, g'_n) and there exists $y \in \mathbb{H}^3$ with $\rho(g'_1 \cdot y, y) < C(n)$.*

In other words, if G acts by isometries on \mathbb{H}^3 , then G is either free or contains a nontrivial element with small translation length. We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose M is a closed 3-manifold with $\mathrm{rank}(\pi_1(M)) = n$, and suppose $L \subset M$ is a hyperbolic link with $\mathrm{esssys}(L) \geq C(n)$, from Theorem 2.2. Now let

$$f : \Gamma \rightarrow M \setminus L$$

be an L -relative 1-spine. Choose any basis (h_1, \dots, h_n) for $\pi_1(\Gamma)$, and we write $g_j = f_*(h_j)$, for each j . By Theorem 2.2

$$\langle g_1, \dots, g_n \rangle = f_*(\pi_1(\Gamma))$$

is free with basis (g_1, \dots, g_n) , or (g_1, \dots, g_n) is Nielsen equivalent to (g'_1, \dots, g'_n) so that for some $y \in \mathbb{H}^3$, we have $\rho(g'_1 \cdot y, y) < C(n)$. First, we suppose we are in the latter case and derive a contradiction.

Let $i : M \setminus L \rightarrow M$ be the inclusion, and note that $i_*(g'_1)$ is necessarily trivial in $\pi_1(M)$ since it represents a loop with length less than $\mathrm{esssys}(L)$. By Lemma 2.1, the Nielsen moves on (g_1, \dots, g_n) lift to Nielsen moves on (h_1, \dots, h_n) , producing a new basis (h'_1, \dots, h'_n) of $\pi_1(\Gamma)$ so that $f_*(h'_j) = g'_j$. Since

$$(i \circ f)_*(h'_1) = i_*(f_*(h'_1)) = i_*(g'_1),$$

we see that

$$\pi_1(M) = i_*(f_*(\pi_1(\Gamma))) = \langle i_*(g'_1), \dots, i_*(g'_n) \rangle = \langle i_*(g'_2), \dots, i_*(g'_n) \rangle,$$

contradicting the fact that $\mathrm{rank}(\pi_1(M)) = n$.

Therefore, $f_*(\pi_1(\Gamma))$ is free on (g_1, \dots, g_n) , and hence f_* is injective by the Hopfian property of free groups. Since $f : \Gamma \rightarrow M \setminus L$ was an arbitrary L -relative 1-spine, it follows that L is a filling link. \square

3. LINKS WITH LARGE ESSENTIAL SYSTOLE

The goal of this section is to prove the following.

Theorem 1.5 *Given a closed, orientable 3-manifold M with $\mathrm{rank}(\pi_1(M)) \geq 1$ and $r > 0$, there exists a hyperbolic link $L \subset M$ such that*

$$\mathrm{esssys}(L) > r.$$

The proof involves constructing a particular tangle in a tetrahedron which, when removed from the tetrahedra (3-simplices) of a triangulation of a 3-manifold, produces a link. We apply this construction to a particularly nice kind of triangulation on M , and show that the resulting link is hyperbolic, and furthermore, if the “combinatorial systole” of the triangulation is large, then the essential systole of the link is large.

3.1. Triangulations. Suppose M is a closed 3-manifold, and τ is a triangulation of M . Let $\hat{\tau}$ denote the subgraph of the 1-skeleton of the first barycentric subdivision of τ which is the union of 1-simplices adjacent to barycenters v_T of the tetrahedra T of τ . The graph $\hat{\tau}$ is bipartite, with the barycenters of tetrahedra one color and all other vertices the second color. We equip $\hat{\tau}$ with the usual graph metric which assigns length 1 to each edges. Then for every two tetrahedra T and T' of τ , the distance between v_T and $v_{T'}$ is 2 if and only if T and T' are distinct and share a vertex, edge, or face.

We define the *combinatorial systole* of $\hat{\tau}$ to be:

$$\text{combsys}(\hat{\tau}) = \min\{\ell_{\hat{\tau}}(\gamma) : \gamma \text{ a loop in } \hat{\tau} \text{ which is not nullhomotopic in } M\}.$$

We will ultimately be interested in manifolds with triangulations with large combinatorial systole. The main consequence of this for our purposes is the following.

Lemma 3.1. *If τ is any triangulation of a closed 3-manifold M , then any non-null-homotopic loop in M has non-empty intersection with least $n = \lfloor \frac{\text{combsys}(\hat{\tau})}{16} \rfloor$ tetrahedra, T_0, \dots, T_{n-1} of τ such that v_{T_i} and v_{T_j} are distance at least 8 apart for all $i \neq j$.*

Proof. First observe that we may homotope γ so that it is a combinatorial loop in the graph $\hat{\tau}$ meeting the exact same set of tetrahedra. We will find the required tetrahedra from those whose barycenters are vertices of γ .

First observe that for the closed ball in $\hat{\tau}$ of radius $R = \lfloor \frac{\text{combsys}(\hat{\tau})}{2} \rfloor - 2$ centered at any vertex of $\hat{\tau}$, the inclusion of this ball into M must be trivial on π_1 . To see this, observe that there is a maximal tree so that every point is connected to the center of the ball by a path of length at most R in the tree, and hence the fundamental group is generated by loops of length at most $2R + 1 < \text{combsys}(\hat{\tau})$. These loops are all trivial by definition of the combinatorial systole, so the image of the fundamental group of the ball is trivial.

Now pick any vertex v_{T_0} on γ which is the barycenter of a tetrahedron T_0 and let n be as in the lemma. Since γ is non-null-homotopic in M , it follows that it is not contained in the closed ball of radius

$$8(n-1) = 8 \left\lfloor \frac{\text{combsys}(\hat{\tau})}{16} \right\rfloor - 8 < R.$$

In particular, γ nontrivially intersects the spheres of radius $8k$ centered at v_{T_0} , where $k = 0, \dots, n-1$, and we let $v_k \in \gamma$ be any vertex of intersection. Since we are considering spheres of even radius, and a vertex in $\hat{\tau}$ whose distance from v_{T_0} is even must also be a barycenter of a tetrahedron, it follows that $v_k = v_{T_k}$ for some tetrahedron T_k for each k . By construction, for all i and j , v_{T_i} and v_{T_j} are distance at least $8|j-i|$ apart, proving the lemma. \square

We are interested in triangulations with large combinatorial systole to which we can apply the previous lemma. We will also want to impose some additional control on the local structure of our triangulations. This additional control can be described by constraints on the links of vertices, which we now describe.

A *Cooper-Thurston triangulation* τ of a 3-manifold is one for which the link of every vertex is isomorphic to the double over the boundary of one of the five triangulations of a disk shown in Figure 1. In [CT88], Cooper and Thurston proved that such triangulations exist. A minor modification of their construction proves the next proposition.

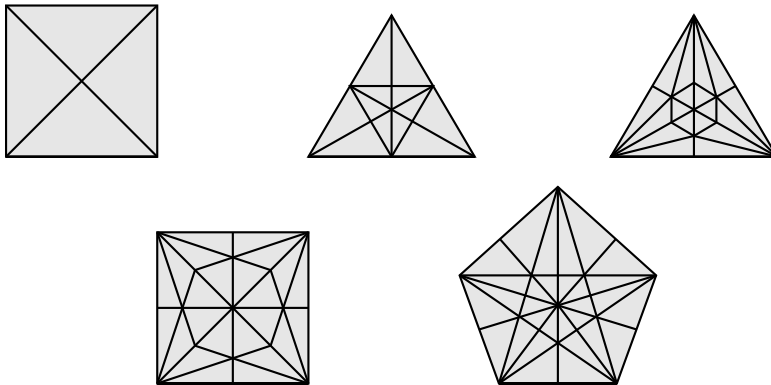


FIGURE 1. The five allowable triangulations of links of vertices are obtained by doubling each of these triangulations of a disk.

Proposition 3.2. *Suppose M is a closed, orientable 3-manifold M and $r > 0$. Then M admits a Cooper-Thurston triangulation τ with $\text{combsys}(\hat{\tau}) > r$.*

Proof. Cooper and Thurston's proof starts with a *paving* of M , which is a subdivision of M into 3-dimensional cubes so that two such cubes either meet at a vertex, edge, or face, or are disjoint. Given an edge e in a cube, the *degree* of e is the number of cubes that have e as an edge. Cooper and Thurston prove there exists a paving of M so that each edge has degree 3, 4, or 5, and the degree 3 and 5 edges form disjoint embedded 1-manifolds. They then triangulate each cube to ensure the correct links (see Figure 2). Note that the triangulation of each cube consists of three kinds of edges: the purple edges, which connect a vertex of the cube to the centers of the three faces in which the vertex is contained; the blue edges, which connect the vertices of the cube to the center of the cube; and the red edges, which connect the center of a face to the center of the cube. Since the resulting triangulation τ depends on the paving P of M , we will denote it $\tau(P)$. As before, $\hat{\tau}(P)$ will be the graph associated with $\tau(P)$ equipped with the length function $\ell_{\hat{\tau}(P)}$ induced by the usual graph metric.

Cooper and Thurston show there is a paving P of M so that the triangulation of each cube as above results in an honest triangulation $\tau(P)$. For any integer $k \geq 1$, we can subdivide each cube of P into k^3 sub-cubes, producing a new paving P_k . Triangulating each of the sub-cubes as above results in another Cooper-Thurston triangulation of M . (In fact, the construction in [CT88] has such a subdivision of the paving built into it.) We will show that for any $r > 0$, we can choose some $k(r) > 0$ sufficiently large so that for the paving $P_{k(r)}$, we have $\text{combsys}(\hat{\tau}(P_{k(r)})) > r$.

Given a paving P for which $\tau(P)$ is a Cooper-Thurston triangulation, we can define a singular, locally Euclidean geodesic metric on M in which each cube of P is (locally) isometric to a unit cube in \mathbb{R}^3 ; that is, a cube where all side lengths equal 1. For any path α in M , we write $\ell_P(\alpha)$ to denote its length in this metric, due to its dependence on P .

Since each edge e of $\hat{\tau}(P)$ is a path in M , we can measure its length, $\ell_P(e)$. Here we derive an upper bound on $\ell_P(e)$ for any edge e of $\hat{\tau}(P)$. For this, we assume the cube is embedded in \mathbb{R}^3 with vertices at the points

$$\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_i \in \{0, 1\} \text{ for } i = 1, 2, 3\}.$$

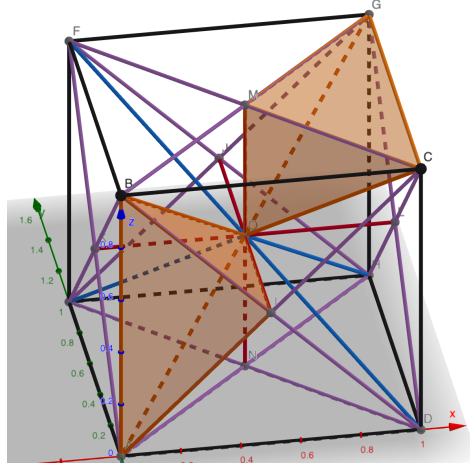


FIGURE 2. An illustration of the Cooper-Thurston triangulation on a single cube. The cube is triangulated by 24 tetrahedra, two of which are shown in orange.

Then the center of C is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and the center of one of the faces is $(\frac{1}{2}, 0, \frac{1}{2})$. One then obtains that the barycenter of the tetrahedon T with vertex set

$$\left\{ (0, 0, 0), (0, 0, 1), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\}$$

is $(\frac{1}{4}, \frac{1}{8}, \frac{1}{2})$. We then have that the maximum distance between a vertex of T and $(\frac{1}{4}, \frac{1}{8}, \frac{1}{2})$ is equal to $\frac{\sqrt{21}}{8}$, which is an upper bound for the edge lengths of $\hat{\tau}(P)$.

When every cube C in a paving P is subdivided into k^3 cubes to produce the new paving P_k , for any curve γ , $\ell_{P_k}(\gamma) = k\ell_P(\gamma)$. We can see this most clearly when γ is an edge of C . In the new metric, s_C , an edge of C , will pass through k cubes each of side length equal to $1 = \ell_P(s_C)$, so $\ell_{P_k}(s_C) = k = k\ell_P(s_C)$.

Consider any paving P so that $\tau(P)$ is a Cooper-Thurston triangulation, as before. Let γ be a non-null-homotopic curve in M which minimizes $\ell_P(\gamma)$ among all such closed curves γ . Find $k(r)$ so that the paving $P_{k(r)}$ obtained by subdividing each cube in P into $(k(r))^3$ cubes will yield $\ell_{P_{k(r)}}(\gamma) > r$, which is made possible by the discussion in the previous paragraph. Observe that γ will still minimize length among non-null-homotopic curves with respect to the new metric induced by $P_{k(r)}$, as the new metric is the old metric scaled by a factor of $k(r)$.

Consider the combinatorial systole of $\hat{\tau}(P_{k(r)})$, which we will call γ' . Note that for each edge $e \in \gamma'$, $\ell_{P_{k(r)}}(e) \leq \frac{\sqrt{21}}{8}$ by construction. Thus,

$$\begin{aligned} \text{combsys}(\hat{\tau}(P_{k(r)})) &= \ell_{\hat{\tau}(P_{k(r)})}(\gamma') = (\# \text{ of edges in } \gamma')(1) \\ &\geq (\# \text{ of edges in } \gamma') \left(\frac{\sqrt{21}}{8} \right) \\ &\geq \sum_{e \in \gamma'} \ell_{P_{k(r)}}(e) \geq \ell_{P_{k(r)}}(\gamma') \\ &\geq \ell_{P_{k(r)}}(\gamma) > r. \end{aligned}$$

The second to last inequality follows since γ minimized length over all non-null-homotopic curves, and γ' is non-null-homotopic. \square

We will also need the following, which is immediate by inspecting Figure 1.

Lemma 3.3. *Suppose τ is a Cooper-Thurston triangulation of a closed 3-manifold M . Then every link of a vertex is flag, and every edge has degree 4, 6, 8, or 10.* \square

Remark 3.4. *Brady-McCammond-Meier [BMM04] describe another construction of triangulations with related bounds on the combinatorics. It seems likely that the barycentric subdivisions of these triangulations could also be made to work for our purposes below, but constructing Cooper-Thurston triangulations with large combinatorial systole is likely easier.*

3.2. A hyperbolic tetrahedron tangle. Let T_0 be a tetrahedron and $L_0 \subset T_0$ be the tangle in T_0 , which is the union of the four embedded circles and properly embedded arcs shown on the left in Figure 3. Explicitly, we assume T_0 is a regular, Euclidean tetrahedron with side lengths 1 and

- (1) each circle has radius $1/4$ and is centered at the barycenter of the face, bounding a disk, and
- (2) each arc is the intersection with T_0 of a circle of radius $1/8$ centered on the barycenter of an edge and contained in a plane orthogonal to the edge.

From the assumptions, the tangle meets each face as shown on the right in Figure 3, and the endpoints of the arcs are inside the disks bounded by the circles in the faces.

Remark 3.5. *Our use of the term “tangle” may be slightly non-standard, but we will only use it in reference to the specific embedded 1-manifold in T_0 .*

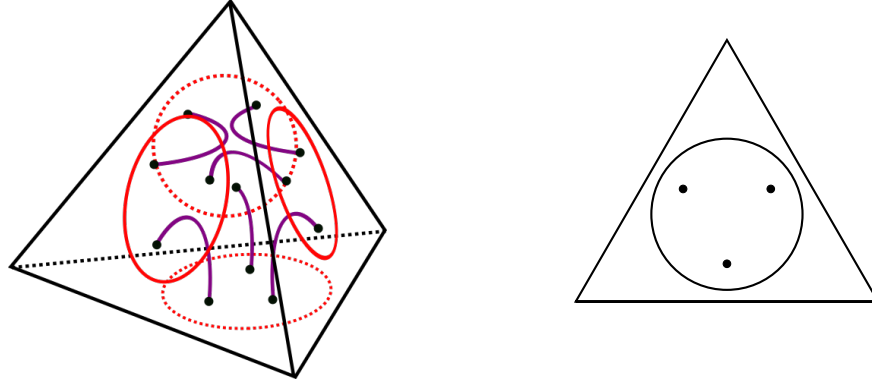


FIGURE 3. **Left:** The tangle $L_0 \subset T_0$. **Right:** The intersection of L_0 with the face.

By construction, the full symmetry group $\mathfrak{Set} \cong S_4$ of T_0 acts on T_0 by isometries preserving L_0 . We can take a fundamental domain, Δ , in T_0 for \mathfrak{Set} to be (any) 2-simplex of the first barycentric subdivision of T_0 , and L_0 meets Δ in a pair of arcs contained in a pair of sides. The fundamental domain Δ and pair of arcs are illustrated in Figure 4.

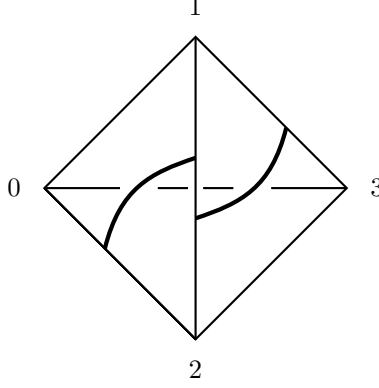


FIGURE 4. The fundamental domain in T_0 for the action of $\mathfrak{T} \text{et}$. The vertices are barycenters of a vertex, edge, face, and of T_0 , and are labeled here by 0, 1, 2, and 3, respectively.

We consider $T_0 \setminus L_0$ as a manifold with corners. The corners are at the 1-skeleton of T_0 , and the boundary consists of the union of the faces minus L_0 , which are each thus homeomorphic to a 2-simplex minus a circle and three points in the disk bounded by the circle, as on the right in Figure 3.

Proposition 3.6. *There is a complete hyperbolic structure on $T_0 \setminus L_0$ of finite volume, such that all boundary components are totally geodesic and all dihedral angles at the corners are $\frac{\pi}{2}$.*

Proof. We construct a hyperbolic structure on the fundamental domain $\Delta \setminus L_0$ explicitly. To do this, we will find a complete, finite volume hyperbolic structure on $\Delta \setminus L_0$ with totally geodesic boundary, whose dihedral angles along the corners are as illustrated on the left of Figure 5. By collapsing the two arcs to points, it suffices to find a partially ideal hyperbolic polyhedron as illustrated on the right of Figure 5, where the two “dots” are ideal (hence deleted).

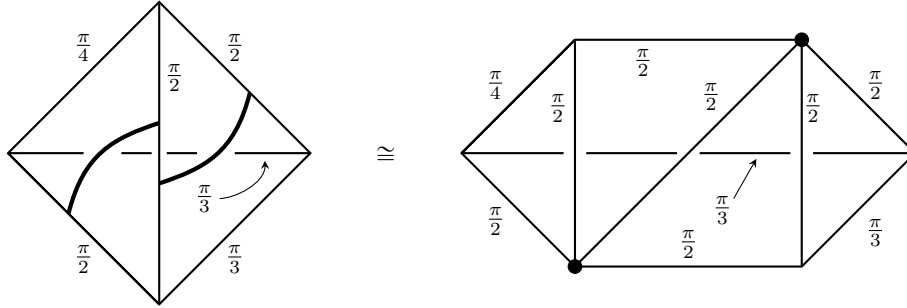


FIGURE 5. **Left:** Dihedral angles for hyperbolic structure on $\Delta \setminus L_0$. **Right:** Partially ideal polyhedron P with ideal vertices obtained by collapsing arcs to a point (illustrated by a dot).

We can construct such a polyhedron $P \subset \mathbb{H}^3$ explicitly as the intersection of hyperbolic half-spaces in the upper half space model $\mathbb{H}^3 = \{(x, y, z) \mid z > 0\}$.

In these coordinates, the polyhedron is given by the set of points $(x, y, z) \in \mathbb{H}^3$ satisfying

- (1) $0 \leq x \leq 3 + \frac{3\sqrt{2}}{2}$,
- (2) $0 \leq y \leq 1 + \frac{\sqrt{2}}{2}$,
- (3) $x^2 + (y - 1)^2 + z^2 \geq 1$, and
- (4) $(x - (2 + \sqrt{2}))^2 + y^2 + z^2 \geq (2 + \sqrt{2})^2$.

One ideal vertex is at ∞ and the other is at $(0, 0, 0)$. Figure 6 shows the polyhedron viewed from the vertex at infinity.

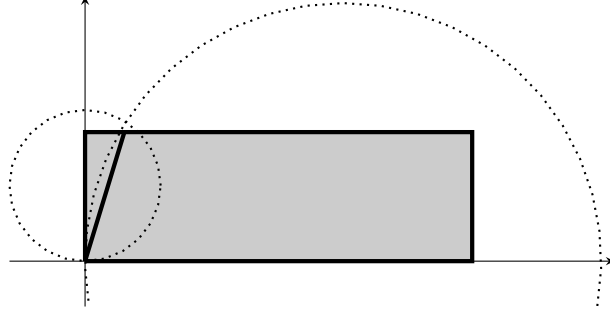


FIGURE 6. View of $P \subset \mathbb{H}^3$ from above.

The hyperbolic half-spaces are bounded by hyperbolic planes that meet the sphere at infinity in four lines and two circles. The equations defining these lines and circles are obtained by making the inequalities above into equations, and setting $z = 0$. One can directly check that the lines and circles intersect in the required angles, and hence so do the hyperbolic planes. \square

3.3. Construction of links and proof of Theorem 1.5. Given a triangulation τ of a closed 3-manifold, assume that each simplex is regular Euclidean with side length 1 and the face gluings are by isometries. The copies of L_0 in each tetrahedron match up to define a link we denote $L_\tau \subset M$. That is, $L_\tau \subset M$ is a link such that for every tetrahedron T in τ , the pair $(T, L_\tau \cap T)$ is homeomorphic to (T_0, L_0) .

The first fact we will need is the following.

Lemma 3.7. *If τ is a Cooper-Thurston triangulation of M , then L_τ is a hyperbolic link.*

Proof. Fix the hyperbolic structure on $T_0 \setminus L_0$ from Proposition 3.6. Then $M \setminus L_\tau$ can be obtained by gluing copies of the hyperbolic structure on $T_0 \setminus L_0$ by isometries along the boundary, defining a finite volume, piecewise hyperbolic structure. Since the dihedral angles of all corners are $\frac{\pi}{2}$, and since the links of every vertex is flag, by Lemma 3.3, it follows that the metric is locally $\text{CAT}(-1)$; see [BH99]. Consequently, the universal cover is $\text{CAT}(-1)$. By Thurston's Hyperbolization Theorem for (the interiors of) compact manifolds with non-empty boundary, it follows that $M \setminus L_\tau$ is hyperbolic; see [Thu86, Mor84, McM92]. \square

Continue to assume that τ is a Cooper-Thurston triangulation. We let \mathcal{V}_τ , \mathcal{E}_τ , \mathcal{F}_τ , and \mathcal{T}_τ denote the set of vertices, edges, faces, and tetrahedra of τ . We now describe a canonical collection of surfaces \mathcal{S}_τ associated to τ . This collection of surfaces is indexed by $\mathcal{E}_\tau \sqcup \mathcal{F}_\tau \sqcup \mathcal{T}_\tau$, as follows:

- (1) For every $F \in \mathcal{F}_\tau$, there is a disk Σ_F bounded by the component of L_τ embedded in F .
- (2) For every $E \in \mathcal{E}_\tau$, there is a component of L_τ that encircles E , built from a subset of the arcs of L_τ intersected with the tetrahedra containing E .
- (3) For every $T \in \mathcal{T}_\tau$, there is a four-holed sphere Σ_T which starts with the boundary of T minus the interior of the four disks $\Sigma_{F_1}, \dots, \Sigma_{F_4}$ contained in the four faces F_1, \dots, F_4 of T and pushing it into T .

See Figure 7.

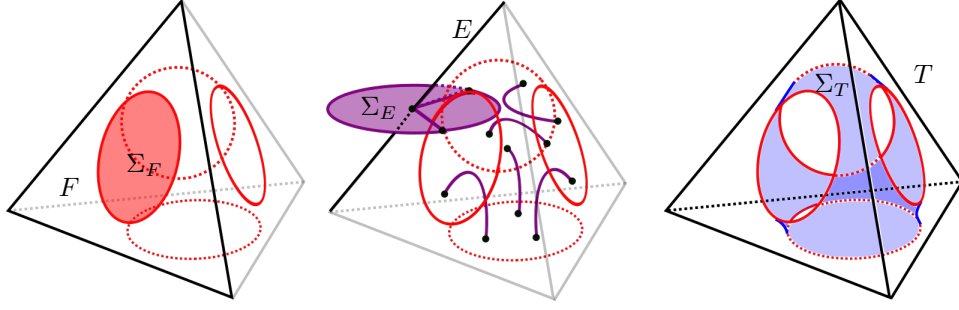


FIGURE 7. **Left:** A face F of a tetrahedron and the associated surface Σ_F . **Middle:** An edge E of a tetrahedron, and the associated surface Σ_E . **Right:** The four-holed sphere Σ_T associated with a tetrahedron T . For clarity, L_τ is only partially shown in the left and right figures.

Let Σ_x° denote the intersection of Σ_x with $M \setminus L_\tau$,

$$\Sigma_x^\circ = \Sigma_x \cap M \setminus L_\tau$$

for all $\Sigma_x \in \mathcal{S}_\tau$. We also write

$$\mathcal{S}_\tau^\circ = \{\Sigma_x^\circ \mid \Sigma_x \in \mathcal{S}_\tau\}.$$

By inspection, we see that each $\Sigma_x^\circ \in \mathcal{S}_\tau^\circ$ is a punctured sphere. More precisely, we have the following.

- (1) For every $F \in \mathcal{F}_\tau$, Σ_F° is a four-punctured sphere.
- (2) For every $E \in \mathcal{E}_\tau$, Σ_E° is a $(k+1)$ -punctured sphere, where k is the degree of the edge E .
- (3) For every $T \in \mathcal{T}_\tau$, Σ_T° is a four punctured sphere.

In particular, the number of punctures is uniformly bounded (at most 11) by Lemma 3.3.

Remark 3.8. We emphasize that, except for $x = T$, Σ_x° is not the interior of Σ_x since there are other components of L_τ that puncture Σ_x . We also note that the surface Σ_x we have described naturally intersects L_τ minimally in the isotopy class, rel $\partial\Sigma_x$ (which can be seen by considerations of the algebraic intersection number, for example).

We begin with some basic properties of these surfaces.

Lemma 3.9. *Suppose τ is a Cooper-Thurston triangulation. Then each $\Sigma_x^\circ \in \mathcal{S}_\tau^\circ$ is totally geodesic in the locally CAT(−1) metric on $M \setminus L_\tau$. In particular, each such Σ_x° is incompressible and quasi-Fuchsian in the hyperbolic structure.*

Recall that a surface Σ with negative Euler characteristic which is properly embedded in a 3-manifold M is *incompressible* if the inclusion $i : \Sigma \hookrightarrow M$ induces an injective map between fundamental groups. An incompressible surface Σ in a hyperbolic 3-manifold is *quasi-Fuchsian* if its fundamental group has a quasi-circle as its limit set for the action on the sphere at infinity of \mathbb{H}^3 .

Proof. We consider Σ_x° for $x \in \mathcal{E}_\tau \cup \mathcal{F}_\tau \cup \mathcal{T}_\tau$ separately, depending on whether x is an edge, face, or tetrahedron.

The statement is clear for $x = F \in \mathcal{F}_\tau$, by construction of the hyperbolic structure on $T_0 \setminus L_0$. For $x = E \in \mathcal{E}_\tau$, note that if we just glue together the tetrahedra around E , then there is an isometric involution fixing Σ_E pointwise, and hence Σ_E° is totally geodesic in this union of tetrahedra, and hence in $M \setminus L_\tau$ (with the locally CAT(−1) metric).

For $x = T \in \mathcal{T}_\tau$, we observe that Σ_T is entirely contained in T , so we may consider the case of $\Sigma_{T_0}^\circ \subset T_0 \setminus L_0$. First we note that $\Sigma_{T_0}^\circ$ is incompressible: the boundary of a compressing disk would necessarily subdivide the four-holed sphere into two pairs of pants, and compressing would produce a pair of annuli between distinct cusps, which is impossible. The orientation preserving subgroup $\mathfrak{T}et$ preserves (the isotopy class of) $\Sigma_{T_0}^\circ$, and the quotient of $\Sigma_{T_0}^\circ$ in $T_0 \setminus L_0$ is an orbifold with one puncture and two cone points of order 2 and 3. In particular, the orbifold must be totally geodesic (c.f. [Ada85]), and hence $\Sigma_{T_0}^\circ$ is totally geodesic.

Being totally geodesic in the locally CAT(−1) metric implies that the surfaces are incompressible. The universal covers of the surfaces are isometrically embedded in the universal cover of $M \setminus L_\tau$ with its CAT(−1) metric. Since the identity on the universal covers is a quasi-isometry with respect to the CAT(−1) metric on the domain and the hyperbolic metric on the range, it follows that the universal covers of the surfaces are quasi-isometrically embedded in \mathbb{H}^3 . In particular, their limit sets are quasi-circles (c.f. [BH99, Theorem III.H.3.9]); hence, the surfaces are quasi-Fuchsian. \square

We record the following.

Corollary 3.10. *The totally geodesic representatives of the surfaces in \mathcal{S}_τ° with respect to the CAT(−1) metric have the following property: For any two distinct $\Sigma_x^\circ, \Sigma_y^\circ \in \mathcal{S}_\tau^\circ$, either Σ_x° and Σ_y° are disjoint, or they cannot be isotoped to be disjoint and $\{x, y\}$ is a pair $\{E, F\}$ with $E \subset F$ or $\{E, T\}$ with $E \subset T$. In the latter case, the surfaces intersect in a single arc.* \square

This corollary asserts that the intersections of the totally geodesic surfaces in CAT(−1) metric in the isotopy classes of the surfaces in \mathcal{S}_τ° intersect in the “obvious” way. We use these representatives to prove the next lemma. Before we do so, we recall that an intersection point x of curve γ with a properly embedded incompressible surface $S \subset N$ in a hyperbolic 3-manifold N is *essential* if there is a lift $\tilde{\gamma} : \mathbb{R} \rightarrow \mathbb{H}^3$ of γ to the universal cover of N that intersects a component \tilde{S} of the preimage of S in a single point \tilde{x} that projects to x . If there is an essential intersection point between γ and S , we say that they *intersect essentially*, and note that in this case, γ and S cannot be homotoped to be disjoint.

Lemma 3.11. *If γ is a loop in $M \setminus L_\tau$ which is non-null-homotopic in M , then γ essentially intersects at least $n = \lfloor \frac{\text{combsys}(\hat{\tau})}{16} \rfloor$ surfaces*

$$\Sigma_{x_0}^\circ, \dots, \Sigma_{x_{n-1}}^\circ \in \mathcal{S}_\tau^\circ.$$

Moreover, for all $i \neq j$, the cusps of $\Sigma_{x_i}^\circ$ and $\Sigma_{x_j}^\circ$ are not contained in any common cusps of $M \setminus L_\tau$.

Proof. First observe that for each surface Σ_x° , the inclusion into M induces the trivial homomorphism since it factors through the inclusion $\Sigma_x^\circ \rightarrow \Sigma_x$, and Σ_x is either a disk or is contained in a tetrahedron of τ .

Suppose γ is a loop in $M \setminus L_\tau$ which is non-null-homotopic in M (hence also in $M \setminus L_\tau$). From the previous paragraph, it follows that γ cannot be homotoped to lie entirely inside any one of the surfaces Σ_x° . Since each component of L_τ is homotopically trivial in M , we see that γ is non-peripheral (i.e. not freely homotopic into a cusp) in $M \setminus L_\tau$. After a homotopy, we may therefore assume that γ is geodesic with respect to the CAT(−1) metric, and we do so.

We also assume that the isotopy class of Σ_x° is represented by a totally geodesic surface with respect to the CAT(−1) metric. Therefore γ intersects each Σ_x° transversely (possibly empty), and every intersection point is essential.

Given a tetrahedron T , consider the family of surfaces $\mathcal{S}_\tau^\circ(T)$ consisting of surfaces Σ_x° for which x is on of the following:

- (1) A face of T ;
- (2) An edge adjacent to a vertex of T ; or
- (3) A tetrahedron T' with $d_\tau(v_T, v_{T'}) \leq 2$.

By inspection, the set of surfaces $\{\Sigma_x \mid \Sigma_x^\circ \in \mathcal{S}_\tau^\circ(T)\}$ have the property that any component of the complement of their union which intersects T is contractible. Thus, if γ intersects T , then it must have an essential intersection with some surface in $\mathcal{S}_\tau^\circ(T)$.

Now Lemma 3.1 implies that γ intersects at least n tetrahedra T_0, \dots, T_{n-1} for which the barycenters v_{T_i} and v_{T_j} are distance at least 8 in $\hat{\tau}$ if $i \neq j$. Let $\Sigma_{x_i}^\circ \in \mathcal{S}_\tau^\circ(T_i)$ be one of the surfaces essentially intersected by γ . The cusps of $\Sigma_{x_i}^\circ$ are contained in cusps of $M \setminus L_\tau$ that correspond to components of L_τ contained in the union of tetrahedra T' with $d_\tau(v_{T_i}, v_{T'}) \leq 2$. Consequently, the cusps of $M \setminus L_\tau$ that contain the cusps of Σ_i and Σ_j are distinct if $i \neq j$. This completes the proof. \square

We are now ready for the proof of the main theorem.

Theorem 1.5 *Given a closed, orientable 3-manifold M with $\text{rank}(\pi_1(M)) \geq 1$ and $r > 0$, there exists a hyperbolic link $L \subset M$ such that*

$$\text{esssys}(L) > r.$$

Proof. By Proposition 3.2, there exists a sequence of Cooper-Thurston triangulations τ_n such that $\text{combsys}(\hat{\tau}_n) \geq 16n$. The theorem is a consequence of the following.

Claim. The essential systoles of L_{τ_n} tend to infinity, or

$$\lim_{n \rightarrow \infty} \text{esssys}(L_{\tau_n}) = \infty.$$

Proof. Let γ_n be closed geodesic in $M \setminus L_{\tau_n}$ which is non-null-homotopic in M , and which realizes the essential systole of L_{τ_n} . For each n consider the surfaces

$$\Sigma_0^\circ(n), \dots, \Sigma_{n-1}^\circ(n) \in \mathcal{S}_{\tau_n}^\circ,$$

from Lemma 3.11 that essentially intersect γ_n . Since the surfaces in $\mathcal{S}_{\tau_n}^\circ$ are quasi-Fuchsian, we may homotope each $\Sigma_i^\circ(n)$ to a *pleated surface*; that is, the inclusion is homotopic to a 1-Lipschitz map of a hyperbolic surface which is totally geodesic in the complementary regions of a geodesic lamination (see [Thu97]). The geodesic γ_n intersects each $\Sigma_i^\circ(n)$ nontrivially in some point $z_i(n)$.

Now we assume $\ell(\gamma_n) = \text{esssys}(L_{\tau_n})$ does not tend to infinity with n and derive a contradiction. This assumption implies that we may pass to a subsequence, and re-index so that for some $R > 0$ we have $\ell(\gamma_n) < R$ for all n .

Recall that for a hyperbolic manifold N and $\epsilon > 0$, the ϵ -thin part of N is the set $N_{(0,\epsilon)} = \{x \in N \mid \text{inrad}(x) < \epsilon\}$, and $N_{[\epsilon,\infty)} = \{x \in N \mid \text{inrad}(x) > \epsilon\}$ is the ϵ -thick part of N . By the Margulis Lemma, there is an ϵ_0 (depending only on the dimension of N) so that if $\epsilon < \epsilon_0$, then $N_{(0,\epsilon)}$ is a disjoint union of horoball cusp regions and collar neighborhoods of geodesics of length less than 2ϵ . See [Thu97] or [BP92] for more details. If N is a hyperbolic 3-manifold and $\epsilon < \epsilon_0$, then the collars of closed geodesics in $N_{(0,\epsilon)}$ are called *Margulis tubes*. For a hyperbolic surface, Σ , and $\epsilon < \epsilon_0$, the diameter of the thick part $\Sigma_{[\epsilon,\infty)}$ is bounded above and below by constants that depend only on ϵ and the topology of Σ .

The rest of the proof of the claim is divided into two case.

Case 1. There is no $\epsilon > 0$ so that $z_i(n)$ is contained in the ϵ -thick part of $\Sigma_i^\circ(n)$ for all i and n .

Passing to a further subsequence if necessary, we can assume that there is some $z_{i_n}(n)$ in the $\frac{1}{n}$ -thin part of $\Sigma_{i_n}^\circ(n)$ for all n and some i_n . Since a pleated surface is a 1-Lipschitz map into $M \setminus L_{\tau_n}$, it maps ϵ -thin parts in each $\Sigma_i^\circ(n)$ to ϵ -thin parts in $M \setminus L_{\tau_n}$. In particular, γ_n enters arbitrarily deep into the Margulis tube around the geodesic representative of a curve in $\Sigma_{i_n}^\circ(n)$; see [BM82, Mey87]. Since $\ell(\gamma_n) \leq R$, γ_n must be entirely contained in this Margulis tube for all n sufficiently large, and is thus homotopic into $\Sigma_{i_n}^\circ(n)$. This is a contradiction since every loop in $\Sigma_{i_n}^\circ(n)$ is null-homotopic in M .

Case 2. There exists $0 < \epsilon < \epsilon_0$ so that $z_i(n)$ is contained in the ϵ -thick part of $\Sigma_i^\circ(n)$ for all i and n .

Without loss of generality, we assume that ϵ is small enough so that distinct ϵ -thin parts of $M \setminus L_{\tau_n}$ are 1-separated. The ϵ -thick part of $\Sigma_i^\circ(n)$ has uniformly bounded diameter since $\Sigma_i^\circ(n)$ has bounded Euler characteristic (it is a sphere with at most 11 punctures). Thus there is a boundary component of the ϵ -thick part of $\Sigma_i^\circ(n)$ within some fixed distance δ from $z_i(n)$ for all i and n . For each i and n , let $w_i(n)$ be a point in the ϵ -thin part of $\Sigma_i^\circ(n)$ which is distance at most $\delta + 1$ from $z_i(n)$.

For each n , we note that the points $w_0(n), \dots, w_{n-1}(n)$ are in distinct thin parts of $M \setminus L_{\tau_n}$. This follows directly from the lemma if the thin parts are horoball cusps, and otherwise it follows by considering the totally geodesic representatives

of the isotopy classes of the surfaces in the $\text{CAT}(-1)$ metric: since the surfaces are pairwise disjoint, no geodesic in one is homotopic to a curve in another.

Since $\{w_0(n), \dots, w_{n-1}(n)\}$ are in distinct ϵ -thin parts, which are 1-separated, this set of points is also 1-separated. Lift γ_n to a geodesic path $\tilde{\gamma}_n$ of length at most R in the universal cover \mathbb{H}^3 based at some point p . For each $z_i(n) \in \gamma_n$, let $\tilde{z}_i(n)$ be a point on $\tilde{\gamma}_n$ that projects to $z_i(n)$ and $\tilde{w}_i(n)$ be a point within distance $\delta + 1$ of $\tilde{z}_i(n)$ that projects to $w_i(n)$. See Figure 8. Then $\{\tilde{w}_0(n), \dots, \tilde{w}_{n-1}(n)\}$ is also a 1-separated set of n points. On the other hand, for each n , these points are contained in a ball of fixed radius $R + \delta + 1$ in \mathbb{H}^3 . This is a contradiction for n sufficiently large since a 1-separated set in such a ball contains at most $\frac{V(R+\delta+3/2)}{V(1/2)}$ points, where $V(r)$ is the volume of a hyperbolic ball of radius r .

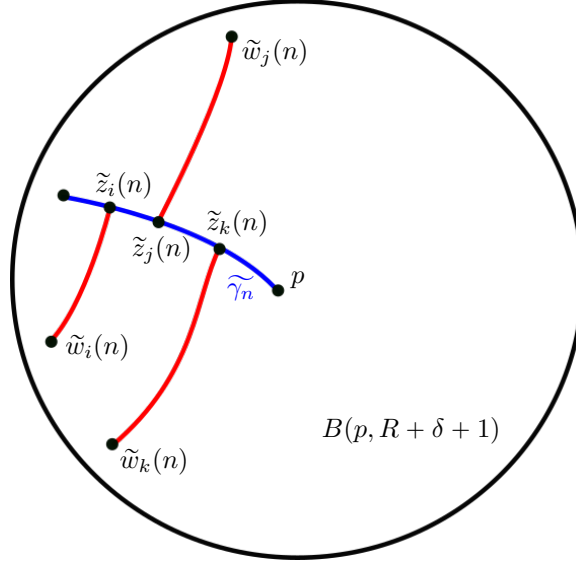


FIGURE 8. The points $\{\tilde{w}_0(n), \tilde{w}_1(n), \dots, \tilde{w}_{n-1}(n)\}$ are all contained in the ball $B(p, R + \delta + 1) \subset \mathbb{H}^3$.

Since a uniform bound on $\text{esssys}(L_{\tau_n})$ for any subsequence produces a contradiction, it follows that

$$\lim_{n \rightarrow \infty} \text{esssys}(L_{\tau_n}) = \infty,$$

as required. □

As already noted, the claim implies the theorem. □

4. CONCLUDING REMARKS

As indicated in the introduction, if one is only interested in proving the links L_τ can be taken to be filling, we can shorten the proof a little, and avoid using Thurston's Hyperbolization Theorem. We now sketch how that can be done. The locally $\text{CAT}(-1)$ metric on $M \setminus L_\tau$ has universal cover that is also 1-hyperbolic

(since triangles are thinner than their comparison triangles in \mathbb{H}^2 which are 1-slim). Consequently, we could apply a Theorem 2.2 to this metric, and deduce the same conclusion in Theorem 1.4 (that is, sufficiently large locally CAT(−1) essential systole implies filling). In this metric, the complement $T \setminus (L_\tau \cap T)$ in any tetrahedron of T of τ has *exactly* the hyperbolic structure constructed in §3.2. Consequently, we can find $\varepsilon > 0$ so that the totally geodesic representatives of disjoint surfaces $\Sigma_x^\circ, \Sigma_y^\circ \in \mathcal{S}_\tau^\circ$ which do not share a cusp in $M \setminus L_\tau$ have disjoint ε -neighborhoods. If ε is small enough, the ε -thin parts are precisely horoball cusp neighborhoods and are ε -separated. Now if γ is a geodesic in this metric realizing the locally CAT(−1) essential systole for L_τ , and n is as in Lemma 3.11, then γ intersects at least n pairwise disjoint surfaces $\Sigma_0^\circ, \dots, \Sigma_{n-1}^\circ$, no two of which share a cusp. Consequently, the length γ is at least $n\varepsilon$, which can be made arbitrarily large.

For any $n \geq 4$, there is a related notion of filling links in closed n -manifolds and an analogue of Question 1.1. Namely, given a smooth, closed n -manifold, M , a 1-spine $f: \Gamma \rightarrow M$ is a minimal rank graph with $f_*: \pi_1(\Gamma) \rightarrow \pi_1(M)$ surjective. A filling link is an embedded, codimension 2 submanifold $i: L \rightarrow M$ such that for any $f: \Gamma \rightarrow M \setminus L$ where $i \circ f$ is a 1-spine, we have that f_* is injective.

This leads us to following question:

Question 4.1. *Which smooth, closed manifolds M with $\dim(M) \geq 4$ contain filling links?*

One approach to construct such links might be to modify the sketch above, at least in some low dimensions and for manifolds admitting nice triangulations. Specifically, can one find some explicit CAT(−1) metric coming from hyperbolic metrics on an n -simplex minus a tangle? We note that one cannot hope to find honest hyperbolic links in general, even for the case of $\dim M = 4$; see [Sar18].

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