A Topologically Rigid Set of Quotients of the Davis Complex

Yandi Wu

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Abstract

A class of topological spaces is *topologically rigid* if any two spaces with the same fundamental group are also homeomorphic. Topological rigidity, in addition to its intrinsic interest, has been useful for solving abstract commensurability questions. In this paper, we explore the topological rigidity of quotients of the Davis complex of certain right angled Coxeter groups by providing conditions on the defining graphs that obstruct topological rigidity. Furthermore, we explore why topological rigidity is hard to achieve for quotients of the Davis complex. Nonetheless, we conclude by introducing infinitely many infinite topologically rigid subclasses.

1 Introduction

Often, determining whether two topological objects are homeomorphic is a significantly harder problem than determining whether their fundamental groups are isomorphic. In some cases, however, if we impose enough conditions on the topological spaces we are studying, the weaker equivalence relation (isomorphism between fundamental groups) implies the stronger and often more useful equivalence relation (homeomorphism between the topological objects). We can often exploit the topological rigidity of such sets of spaces to derive useful results (recall that a collection of topological objects $\mathscr{X}$ is *topologically rigid* if for any $X_1, X_2 \in \mathscr{X}$, if $\pi_1(X_1) \cong \pi_1(X_2)$, then X_1 and X_2 are homeomorphic). For example, to determine if two groups G_1 and G_2 are *abstractly commensurable* (i.e. have isomorphic finite-index subgroups), we often construct two finite-sheeted homeomorphic covers of X_1 and X_2, where $\pi_1(X_i) = G_i$ for $i = 1, 2$. These homeomorphic covers can be hard to construct. However, if the finite-sheeted covers $\tilde{X}_1$ and $\tilde{X}_2$ belong to a topologically rigid class of spaces and $\pi_1(\tilde{X}_1) \cong \pi_1(\tilde{X}_2)$, then we know $\tilde{X}_1$ and $\tilde{X}_2$ are homeomorphic.

There are several well-established examples of topologically rigid classes. For example, the set of closed orientable 2-manifolds is topologically rigid. The Poincare Conjecture implies the set of simply-connected, closed 3-manifolds is topologically rigid. In a series of papers (see [4], [5], and [6]), Lafont proves the set of of simple, thick n-dimensional hyperbolic P-manifolds, a subclass of piecewise CAT(-1) spaces, is topologically rigid for $n \geq 2$. In this paper, we consider certain *orbicomplexes*, unions of collections of orbifolds identified along homeomorphic suborbifolds, associated with Right-Angled Coxeter Groups (RACGs), defined below.

Definition 1.1 (Right-Angled Coxeter Group)**.** Given a finite simplicial graph Γ with edge set E and vertex set V, the *Right-Angled Coxeter Group (RACG)* W_Γ with defining graph Γ is the group with presentation $\langle v_i \in V : v_i^2 = 1, [v_i, v_j] = 1 \text{ if } [v_i, v_j] \in E \rangle$.

A RACG W_Γ acts properly discontinuously by isometeries on a space called the *Davis Complex* Σ_Γ. The quotient $\mathscr{D}_\Gamma = \Sigma_\Gamma / W_\Gamma$, which we call a *Davis orbicomplex*, is one of the aforementioned orbicomplexes and comes equipped with cell stabilizer data defined by the action of W_Γ on Σ_Γ. To clarify, recall that if an amalgamated free product or HNN extension G acts on a Basse Serre tree T, the resulting quotient T/G is a graph of groups whose vertices and edges are labelled by subgroups of G isomorphic to vertex and edge stabilizers of T. Similarly, each edge and vertex of a Davis orbicomplex $\mathscr{D}_\Gamma$ may be labeled by a subgroup of W_Γ that stabilizes a lift of the edge or vertex in Σ_Γ (in this paper, we do not specify such labels as they are not crucial for our proofs). For further background on the Davis complex and Coxeter groups, refer to [3]. The Davis orbicomplex has been studied extensively by Stark, who poses the following question in [7], which we will give a partial answer to in this paper:

Question 1.2. *For which set $\mathscr{W}$ of Coxeter groups is the set of Davis orbicomplexes $\mathscr{D}_\Gamma$ for groups in $\mathscr{W}$ together with their finite-sheeted covers topologically rigid?*

Despite the simplicity of the problem statement, the answer to Question 1.2 is very nuanced. In this paper, we focus our attention on RACGs that are one-ended (Γ has no separating edges or vertices and is connected) and hyperbolic (Γ is square-free, or has no cycles of length four). One example of a class of defining graphs that gives rise to W_Γ satisfying these conditions is a subclass of *generalized* Θ *graphs*, defined as follows:

Definition 1.3 (Generalized Θ-graph)**.** For $k \geq 1$, $0 \leq n_1 \leq \ldots \leq n_k$, let $\Theta = \Theta(n_1, n_2, \ldots, n_k)$ be the graph with two vertices a and b, each of valence k, and k edges $e_1, e_2, \ldots, e_k$ connecting them, which we will call the branches of Θ. Furthermore, for $1 \leq i \leq k$, e_i is subdivided into $n_i + 1$ edges by inserting n_i new vertices.

For the purposes of this paper, we will require that $n_i > 0$ for all $1 \leq i \leq k$ and $n_2 > 1$ in order to ensure W_Γ is hyperbolic.

Associated to each generalized Θ-graph is an Euler characteristic vector, which captures the Euler characteristics of the orbifolds in the Davis orbicomplex $\mathscr{D}_\Gamma$. The Euler characteristic vector is often used to classify Davis orbicomplexes; in [2], the Euler characteristic vector is used to list abstract commensurability criteria. In this paper, we will use Euler characteristic vectors to list criteria for topological rigidity.

Definition 1.4 (Euler characteristic vectors of generalized Θ-graphs)**.** Let $\Theta = \Theta(n_1, n_2, \ldots, n_k)$ be a generalized Θ graph. Then the *Euler characteristic vector* of Θ is the vector $v = (x_1, x_2, \ldots, x_n)$, where $x_i = \frac{1-n_i}{4}$. Two Euler characteristic vectors v_1 and v_2 are said to be *commensurable* if there exist $K, L \in \mathbb{Z}_{\neq 0}$ such that $Kv = Lw$.

Dani, Stark, and Thomas show in Theorem 5.2 of [2] that finite covers of Davis orbicomplexes with $\Gamma = \Theta(n_1, n_2, \ldots, n_k)$ are topologically rigid. In this paper, we focus on cycles of generalized Θ graphs introduced in [2], which consist of generalized Θ graphs identified along their essential vertices.

Definition 1.5 (Cycle of generalized Θ-graphs)**.** Let $N \geq 3$ and let $b_1, b_2, \ldots, b_N$ be positive integers so that for each i, $1 \leq i \leq N$, at most one of b_i and b_{i+1} where i is taken $\mod N$ is equal to 1. Let Θ_i be a generalized Θ graph with b_i edges between two vertices a_i and c_i. We can construct a cycle of N generalized Θ-graphs Γ by identifying c_i with a_{i+1}.

We call the vertex of a cycle of generalized Θ graphs with valence greater than two an *essential vertex*. For the rest of the paper, we will use $\{v_i\}_{i=1}^N$ to denote the set of essential vertices of all the graphs involved. The indices of all v_i's will also taken mod N, where N is the number of essential vertices (or equivalently generalized Θ graphs) in a cycle of generalized Θ-graphs Γ.

Let Γ be a cycle of generalized Θ graphs with Davis complex Σ_Γ, and G a finite index, torsion-free subgroup of W_Γ. Stark proves in [7] that the set of quotients Σ_Γ/G, which correspond to finite-sheeted covers of the Davis orbicomplexes $\mathscr{D}_\Gamma$, is not topologically rigid by constructing $X_1 = \Sigma_\Gamma/G_1$ and $X_2 = \Sigma_\Gamma/G_2$ that are homotopic but not homeomorphic. Theorem 1.7 in Section 2 generalizes the construction from [7] to create a class of orbicomplexes where topological rigidity fails. Our construction of homotopic but not homeomorphic covers relies on the fact that one set of orbifolds in the orbicomplex is a finite cover of another set of orbifolds.

Definition 1.6. Suppose Γ is a cycle of generalized Θ graphs with essential vertices $\{v_i\}_{i=1}^N$, and there exist two essential vertices v_j, v_k such that the generalized Θ graphs Θ_j and Θ_k between v_j and $v_{j+1} \mod N$ and v_k and $v_{k+1} \mod N$ (where $k \neq j$) respectively have commensurable Euler characteristic vectors u and w ($Ku = Lw$ for some $K, L \in \mathbb{Z}_{\neq 0}$). Then we say Γ is *repetitive*. If K or $L = 1$, then we say Γ is *strongly repetitive*.

Theorem 1.7. *Suppose a class of finite-sheeted covers of Davis orbicomplexes $\mathscr{X}$ contains all the finite-sheeted covers of some Davis orbicomplex $\mathscr{D}_\Gamma$ where Γ is strongly repetitive. Then $\mathscr{X}$ is not topologically rigid.*

It is not known whether Theorem 1.7 is true if we only assume Γ is repetitive.

Theorem 1.7 as well as Stark's proof in [7] rely on constructions of finite-sheeted covers of the same Davis orbicomplex $\mathscr{D}_\Gamma$. One can also prove, however, that two finite-sheeted covers of nonhomeomorphic Davis orbicomplexes can also violate topological rigidity.

Definition 1.8 (Permuted pairs)**.** Two cycles of generalized Θ graphs Γ_1 and Γ_2 form a *permuted pair* if they are obtained from identifying the essential vertices of the same set of generalized Θ graphs. Equivalently, the set of Euler characteristic vectors of Γ_1 is some permutation of the set of Euler characteristic vectors of Γ_2.

Remark 1.9. Note that if we use the definition above, it is possible for a permuted pair Γ_1 and Γ_2 to be isomorphic. For example, if Γ_1 and Γ_2 each consist of three generalized Θ graphs glued together, they are isomorphic (see the proof of Lemma 3.4 for details). We do not consider such pairs in Theorem 1.10, stated below.

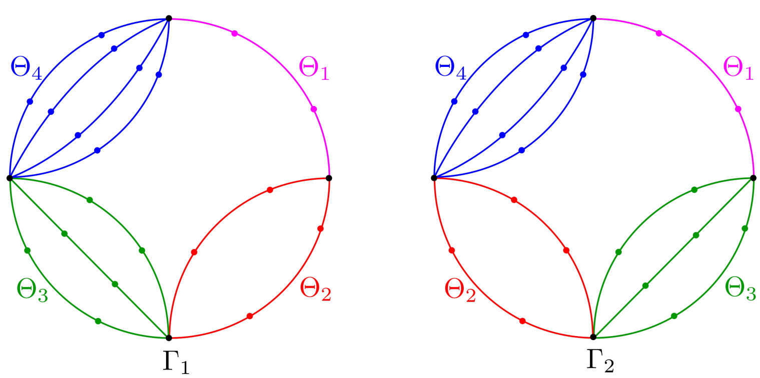

Figure 1: An example of a permuted pair. Note that Γ_1 and Γ_2 both consist of Θ_i (where $1 \leq i \leq 4$) glued along essential vertices.

Theorem 1.10. *Suppose a class of finite-sheeted covers of Davis orbicomplexes $\mathscr{X}'$ contains all finite sheeted covers of two Davis orbicomplexes $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$, where Γ_1 and Γ_2 form a permuted pair and Γ_1 and Γ_2 are not isomorphic. Then $\mathscr{X}'$ is not topologically rigid.*

In the proof of Theorem 1.10, we find two homotopic finite-sheeted covers of $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$ that are not homemomorphic. As a side note, this means that W_{Γ_1} and W_{Γ_2} are commensurable, so having two defining graphs that form a permuted pair is a sufficient condition for commensurability. Recall that in Theorem 1.12 of [2], Dani, Stark, and Thomas provide two necessary and sufficient conditions for commensurability of RACGs with defining graphs that are cycles of generalized Θ graphs.

In [7], Stark constructs X_1 and X_2, two homotopic finite covers of a Davis orbicomplex $\mathscr{D}_\Gamma$ with non-homeomorphic *singular sets* (e.g. sets along which the orbifolds are identified). In her example, Γ is a cycle of generalized Θ graphs, proving that the set of finite-sheeted covers of Davis complexes with defining graphs that are cycles of generalized Θ graphs is not topologically rigid. In light of these results, in Section 3 of [2], Dani, Stark, and Thomas construct a different set of orbicomplexes that is topologically rigid, which they use to prove abstract commensurability results. Nevertheless, in section 4 (see Theorem 4.7), we are able to find a topologically rigid subclass of finite-sheeted covers of Davis orbicomplexes $\mathscr{D}_\Gamma$ where Γ is a cycle of generalized Θ graphs. The subclass also takes Theorems 1.7 and 1.10 into account to exclude finite-sheeted covers of $\mathscr{D}_\Gamma$ that violate topological rigidity. Although the exact statement of the theorem is fairly technical, we state a simplified version below.

Theorem 1.11. *There exists an infinite class $\mathscr{C}$ of Davis orbicomplexes such that for any $D_\Gamma \in \mathscr{C}$, an infinite collection of finite-sheeted covers of D_Γ form a topologically rigid set.*

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2 Preliminaries

We now introduce a construction of the Davis orbicomplex specific to the setting where the defining graph Γ is a cycle of generalized Θ graphs consisting of $\Theta_i = \Theta(n_{i,1}, n_{i,2}, ..., n_{i,k})$ for $1 \leq i \leq N$. For a more detailed construction of W_Γ and verification that $\pi_1(\mathscr{D}_\Gamma)$ is indeed W_Γ, we refer the reader to Section 2 of [7] and Section 3 of [2].

First, we describe how to construct an orbifold $\mathscr{P}_{i,j}$ for a branch (edge) $b_{i,j}$ of a generalized Θ graph Θ_i. For each $b_{i,j}$, construct a $(n_{i,j}+2)$-gon with an edge of order 1, which we call a *nonreflection edge*, and $n_{i,j}+1$ reflection edges of order 2. All the vertices are order 4 vertices, with the exception of the two order 2 vertices adjacent to the non-reflection edge.

Construction 2.1 (Davis Orbicomplex $\mathscr{D}_\Gamma$ of a cycle of generalized Θ graphs)**.** First, we will construct an orbifold graph S. The underlying graph of S is a star with one central vertex v_0 adjacent to N valence one vertices. The valence one vertices are orbifold points of order 2. Cyclically label the orbifold points with v_l where $1 \leq l \leq N$, and use e_l to denote the edge $[v_0, v_l] \in E(S)$. Then attach the set of branch orbifolds $\mathscr{P}_{i,j}$ along their non-reflection edges to e_i and e_{i+1}, where the labels are taken mod N. An example of Construction 2.1 is shown in Figure 2. o

Note that for the cycle of generalized Θ graphs Ψ shown in Figure 2, the Euler characteristic vectors of Θ_1 and Θ_3 are $(-\frac{1}{4}, -\frac{1}{4})$ and $(-\frac{3}{4}, -\frac{3}{4})$, so $3w_1 = w_3$, which means Ψ is strongly repetitive. Theorem 1.7 then implies any class $\mathscr{X}$ that contains all finite-sheeted covers of $\mathscr{D}_\Psi$ is not topologically rigid.

All finite-sheeted covers of Davis orbicomplexes that we construct will contain *jester hats*, a specific kind of orbifold defined below:

Definition 2.2 (Jester hats)**.** Suppose $\mathscr{O} = D^2 \underbrace{(2, 2, ..., 2)}_{n}$, i.e. a disk with n order 2 points. Then we will call $\mathscr{O}$ a jester hat with n (order two) cone points.

3 Examples of topologically non-rigid sets

We first introduce the construction of jester hats that cover orbifolds in a Davis orbicomplex $\mathscr{D}_\Gamma$.

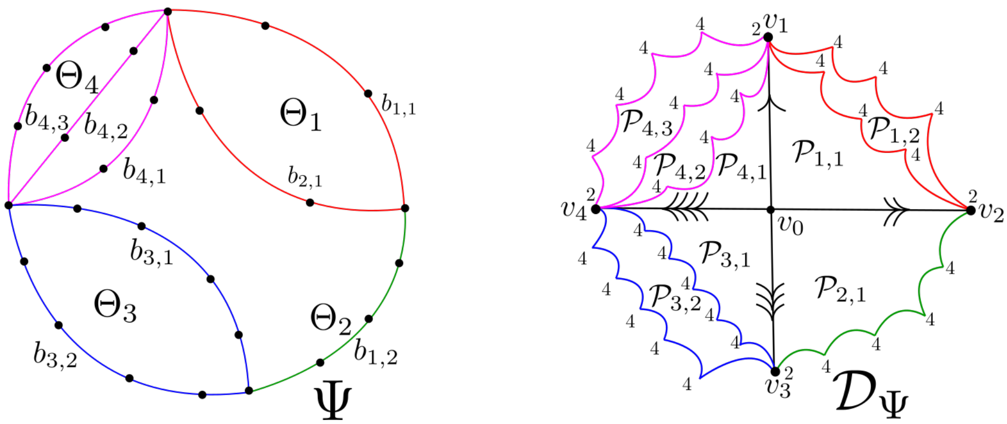

Figure 2: A cycle of generalized Θ graphs Ψ along with its Davis orbicomplex $\mathscr{D}_\Psi$. We label edges e_l in the singular star S with l arrows. Note that each branch $b_{i,j} \in \Psi$ determines an orbifold $\mathscr{P}_{i,j} \in \mathscr{D}_\Psi$.

Lemma 3.1. *Suppose d is a positive even integer. Each orbifold $\mathscr{P}$ with r reflection edges is covered by the jester hat $D^2\underbrace{(2,2,...,2)}_{c}$, where $c = \frac{d}{2}(r-3)+2$ and d is the degree of the cover.*

Proof. This construction is based on Stark's construction in Lemma 3.1 from [7] and Crisp and Paoluzzi's construction from Section 3.1 of [1]. Using Crisp and Paoluzzi's construction, we observe that for any even integer $d > 0$, an orbifold $\widehat{\mathscr{O}}$ with $\frac{d}{2}(r-3)+3$ reflection edges is tiled by $\frac{d}{2}$ copies of an orbifold with r reflection edges, so $\widehat{\mathscr{O}}$ is a $\frac{d}{2}$-sheeted orbifold cover of $\mathscr{O}$. For example, in Figure 3, an orbifold $\widehat{\mathscr{O}}$ with 6 reflection edges is tiled by 3 copies of $\mathscr{O}$, an orbifold with 4 reflection edges. Thus, $\widehat{\mathscr{O}}$ is a 3-sheeted orbifold cover of $\mathscr{O}$. Next, if we unfold along the reflection edges of $\widehat{\mathscr{O}}$, we obtain a closed disk with $\frac{d}{2}(r-3)+2$ order two cone points, as desired. The construction is illustrated in Figure 3.

□

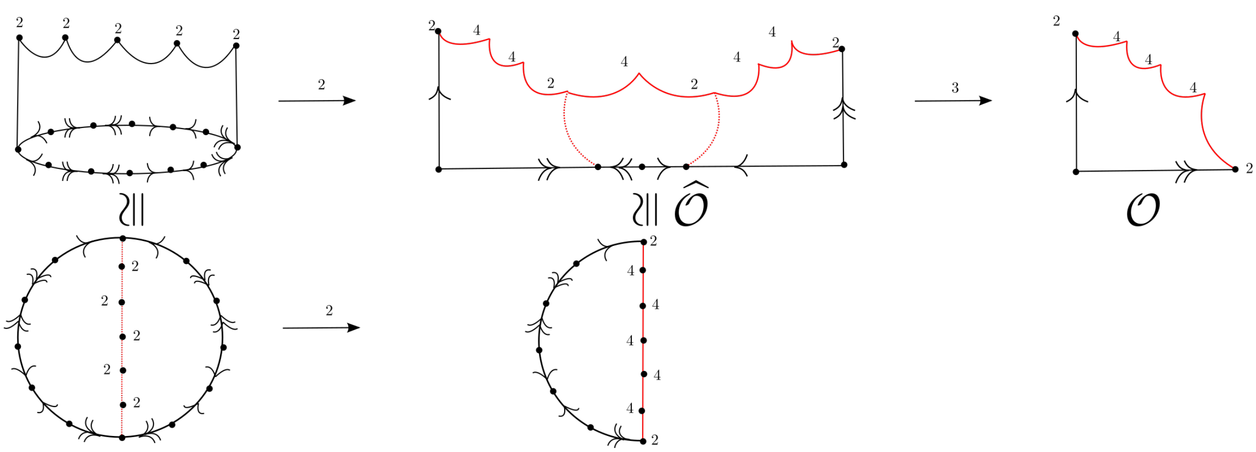

Figure 3: A tower of covers illustrating the lemma. Here, $D^2(2,2,2,2,2)$ is a six-fold cover of an orbifold with four reflection edges.

Construction 3.2 (The double of a singular set)**.** The Davis orbicomplex $\mathscr{D}_\Gamma$ has a singular subset S consisting of an orbifold star graph with N order two points, where N is the number of essential vertices in Γ.

We can construct a double cover of S, which we call $\widehat{S}$, by unfolding along order two points to obtain a subdivided generalized Θ-graph with N branches and one vertex on each branch between the essential vertices. For an illustration, refer to Figure 4. All the singular sets constructed in this section will be a finite-sheeted cover of $\widehat{S}$.

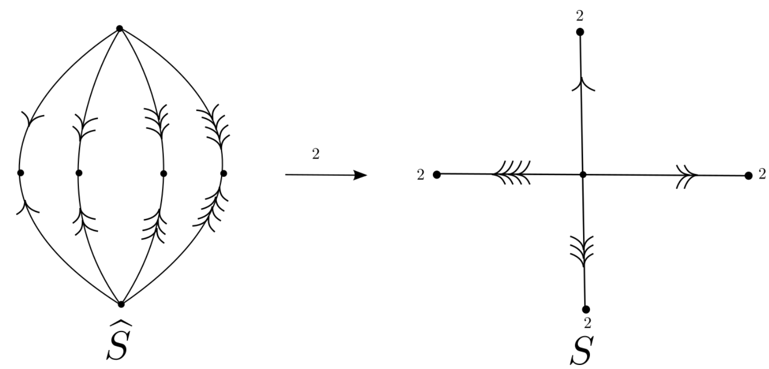

Figure 4: A generalized Θ graph with four branches two-fold covers the singular subset of the orbicomplex from Figure 2. Here, $N = 4$.

For all of the covers described in this paper, all the edges will be subdivided by a copy of the lift of an order two point $\tilde{v}_i$ from the Davis orbicomplex. Both subdivisions will be oriented towards $\tilde{v}_i$ and labeled with the label of the edge, which we will specify in the construction. For simplicity, we will count a subdivided edge as one edge when calculating cycle lengths.

Proposition 3.3. *Let Γ be strongly repetitive. Then there exist homotopic but non-homeomorphic finite-sheeted covers of $\mathscr{D}_\Gamma$.*

Proof. Suppose u and w are Euler characteristic vectors of Θ_i and Θ_k respectively where $Ku = w$ for some $K \in \mathbb{Z}_+$. Without loss of generality, assume $i = 1$ since we can rotate the labels of the essential vertices otherwise. Suppose Θ_1 and Θ_k have l branches. Let $n_{s,b}$ denote the number of vertices on the bth branch of Θ_s. Then if $u = (\frac{1-n_{1,1}}{4}, \frac{1-n_{1,2}}{4}, ..., \frac{1-n_{1,l}}{4})$ and $w = (\frac{1-n_{k,1}}{4}, \frac{1-n_{k,2}}{4}, ..., \frac{1-n_{k,l}}{4})$ then for $1 \leq b \leq l$, $K\big(\frac{1-n_{1,b}}{4}\big) = \big(\frac{1-n_{k,b}}{4}\big)$ and $K(1 - n_{1,b}) = 1 - n_{k,b}$. If $r_{s,b}$ denotes the number of reflection edges on the orbifold in the Davis orbicomplex constructed from the bth branch of Θ_s, then $r_{s,b} = n_{s,b} + 2$, so we have $K(r_{1,b} - 3) + 3 = r_{k,b}$.

Case 1 ($\mathbf{K = 1}$): We will first consider some general cases before addressing the edge case where $N \neq 3$. First, suppose $K = 1$ and $k < N - 1$. Then $r_{1,b} = r_{k,b}$ for $1 \leq b \leq l$. We will construct two non-homeomorphic but homotopic four-sheeted covers of $\mathscr{D}_\Gamma$, which we will call $\widetilde{X_1}$ and $\widetilde{X_2}$. First, we construct their singular sets $\widetilde{S_1}$ and $\widetilde{S_2}$ with four essential vertices, $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$ and $\tilde{v}'_1, \tilde{v}'_2, \tilde{v}'_3, \tilde{v}'_4$. To construct $\widetilde{S_1}$, add k edges between two pairs of vertices: $\tilde{v}_1$ and $\tilde{v}_4$, as well as $\tilde{v}_2$ and $\tilde{v}_3$. The edges will be labeled with all integers between 1 and k and subdivided as described earlier in the section. Between two other pairs of vertices, $\tilde{v}_1$ and $\tilde{v}_2$, as well as $\tilde{v}_3$ and $\tilde{v}_4$, construct $N - k$ subdivided edges labeled with all integers between $k + 1$ and N. Thus, in total, there are two (subdivided) four-cycles labeled with k and $k + 1$ as well as 1 and N, and $N - 2$ cycles of length two labeled with i and $i + 1$, where $1 \leq i \leq N - 1$, $i \neq k$. For the other singular set, $\widetilde{S_2}$, there is one edge labeled with 1 between two pairs of vertices: $\tilde{v}'_1$ and $\tilde{v}'_4$, as well as $\tilde{v}'_2$ and $\tilde{v}'_3$. Additionally, there are $N - 1$ edges between two other pairs of vertices, $\tilde{v}'_1$ and $\tilde{v}'_2$ as well as $\tilde{v}'_3$ and $\tilde{v}'_4$; these edges are labeled with all integers between 2 and N. In total, there are again two four-cycles with edges labeled with 1 and 2 as well as 1 and N, and $N - 2$ cycles of length two labeled with i and $i + 1$ where $2 \leq i \leq N - 1$.

By the conditions imposed on $\mathscr{D}_\Gamma$, the four-cycles labeled by k and $k + 1$ in $\widetilde{S_1}$ and 1 and 2 in $\widetilde{S_2}$ have the same set of jester hats glued to them. Additionally, the two-cycles labeled with 1 and 2 in $\widetilde{S_1}$ and k and

$k+1$ in $\widetilde{S_2}$ have the same set of jester hats glued to them. For other integers I where $1 \le I \le N$ and $I \ne 1, k$, for every cycle in $\widetilde{S_1}$ labeled with I and $I+1$, there is a corresponding cycle of the same length labeled with I and $I+1$ in $\widetilde{S_2}$. As a result, for each cycle in $\widetilde{S_1}$ with a set of jester hats glued to it, there is a corresponding cycle in $\widetilde{S_2}$ with the same set of jester hats glued to it. By taking their regular neighborhoods, we can see that both $\widetilde{S_1}$ and $\widetilde{S_2}$ are homotopic to a $(2N-4)$-holed sphere. We can then conclude that $\widetilde{X_1}$ and $\widetilde{X_2}$ are homotopic to the same $(2N-4)$-holed sphere with the same same sets of jester hats glued to their boundary components. However, they are not homeomorphic since the complements of their cut pairs have different numbers of connected components.

If $N \ne 3$, there are two special cases where the above construction does not work. First, note that if $k = N$, the construction does not work because the first cover will be disconnected. Thus, we construct modified non-homeomorphic but homotopic degree four covers. As before, we will call these covers $\widetilde{X_1}$ and $\widetilde{X_2}$ with singular sets $\widetilde{S_1}$ and $\widetilde{S_2}$ that have essential vertices $\tilde{v_i}$ and $\tilde{v_i'}$ respectively, where $1 \le i \le 4$. For $\widetilde{S_1}$, construct two subdivided edges labeled with 1 and 2 between $\tilde{v_1}$ and $\tilde{v_4}$ as well as $\tilde{v_2}$ and $\tilde{v_3}$. Then between vertices $\tilde{v_3}$ and $\tilde{v_4}$ as well as $\tilde{v_1}$ and $\tilde{v_2}$, construct $N-2$ edges with labels between 3 and N. For $\widetilde{S_2}$, between $\tilde{v_1'}$ and $\tilde{v_4'}$ as well as $\tilde{v_2}$' and $\tilde{v_3'}$, construct $N-1$ subdivided edges labeled with all integers between 3 and N and one subdivided edge labeled with 1. Then construct a subdivided vertex labelled with 2 between $\tilde{v_3'}$ and $\tilde{v_4'}$ as well as $\tilde{v_1'}$ and $\tilde{v_2'}$.

Second, note that if $k = N-1$, the general construction does not work since $\widetilde{S_1}$ and $\widetilde{S_2}$ are homeomorphic; therefore, $\widetilde{X_1}$ and $\widetilde{X_2}$ are also homeomorphic. In order to fix this, construct new $\widetilde{S_1}$ and $\widetilde{S_2}$ as follows: for $\widetilde{S_1}$, construct 3 edges between $\tilde{v_1}$ and $\tilde{v_2}$ as well as $\tilde{v_3}$ and $\tilde{v_4}$, which are labeled with $1, 2$, and N. Then construct $N-3$ edges between $\tilde{v_1'}$ and $\tilde{v_4'}$ as well as $\tilde{v_2'}$ and $\tilde{v_3'}$ labeled from 3 to $N-1$. For $\widetilde{S_2}$, follow the construction for the case where $k = N$. Again, in both edge cases ($k = N$, $k = N-1$, $N > 3$), both $\widetilde{S_1}$ and $\widetilde{S_2}$ are homotopic to a $(2N-4)$-holed sphere and have the same sets of jester hats glued to them, but complements of their cut pairs have different numbers of connected components. Thus, $\widetilde{S_2}$ and $\widetilde{S_1}$ are not homeomorphic but $\widetilde{X_1}$ and $\widetilde{X_2}$ are homotopic.

For the case $N = 3$, any of the previous constructions will yield homeomorphic $\widetilde{X_1}$ and $\widetilde{X_2}$, so a different pair of covers is necessary. For this special case, we will construct 16-sheeted covers. Assume without loss of generality that Θ_1 and Θ_3 have the same Euler characteristic vectors. For both $\widetilde{S_1}$ and $\widetilde{S_2}$, label edges $[v_i, v_{i+1}]$ and $[v_i', v_{i+1}']$ respectively with 2 if i is odd and 3 if i is even. Then construct the edges $[\widetilde{v_2}, \widetilde{v_{13}}]$, $[\widetilde{v_3}, \widetilde{v_{12}}]$, $[\widetilde{v_4}, \widetilde{v_{11}}]$, $[\widetilde{v_1}, \widetilde{v_{16}}]$, $[\widetilde{v_{14}}, \widetilde{v_{15}}]$, $[\widetilde{v_9}, \widetilde{v_{10}}]$, $\widetilde{v_7}, \widetilde{v_8}]$, and $[\widetilde{v_5}, \widetilde{v_6}]$ labelled with 1. For $\widetilde{S_2}$, construct edges $[\widetilde{v_1'}, \widetilde{v_{16}'}]$, $[\widetilde{v_2'}, \widetilde{v_{15}'}]$, $[\widetilde{v_{13}'}, \widetilde{v_{14}'}]$, $[\widetilde{v_3'}, \widetilde{v_{12}'}]$, $[\widetilde{v_4'}, \widetilde{v_5'}]$, $[\widetilde{v_6'}, \widetilde{v_7'}]$, $[\widetilde{v_8'}, \widetilde{v_{11}'}]$, and $[\widetilde{v_9'}, \widetilde{v_{10}'}]$ and label them with 1. As a result, $\widetilde{S_1}$ has one 6-cycle, one 4-cycle, and three 2-cycles labelled with 1 and 2 and one 8-cycle, one 4-cycle, and two 2-cycles labelled with 1 and 3. On the other hand, $\widetilde{S_2}$ has the same set of cycle counts with different labels: one 6-cycle, one 4-cycle, and three 2-cycles labelled with 1 and 3 and one 8-cycle, one 4-cycle, and two 2-cycles labelled with 1 and 2. Thus, both $\widetilde{S_1}$ and $\widetilde{S_2}$ are homotopic to 10-holed spheres, and by construction have the same sets of jester hats glued to them; thus, $\widetilde{X_1}$ and $\widetilde{X_2}$ are homotopic. Observe, however, that $\widetilde{S_1}$ and $\widetilde{S_2}$ are not homeomorphic, as desired. This completes the proof for $K = 1$.

Case 2 (K > 1): Suppose $K > 1$ and $K = p_1^{u_1} p_2^{u_2} \dots p_t^{u_t} = \prod_{d=1}^{t} p_d^{u_d}$ is the prime factorization of K. We will now construct two non-homeomorphic $2\prod_{d=1}^{t} p_d^{u_d+1}$-sheeted covers $\widetilde{X_1}$ and $\widetilde{X_2}$. First, we construct their singular sets $\widetilde{S_1}$ and $\widetilde{S_2}$, which will both have $2\prod_{d=1}^{n} p_d^{u_d+1}$ essential vertices, which we will label $\tilde{v_i}$ and $\widetilde{v'}_i$ for $1 \le i \le 2\prod_{d=1}^{t} p_d^{u_d+1}$. Note that all the vertex indices in the construction will be taken modulo $2\prod_{d=1}^{t} p_d^{u_d}$.

Construction of $\widetilde{S_1}$: First, suppose that $k \ne N$; if $k = N$, the following construction will be disconnected.

For the first singular set $\widetilde{S_1} \subset \widetilde{X_1}$, for even i, construct k edges between $\tilde{v}_i$ and $\tilde{v}_{i+1} \mod 2\prod_{d=1}^{t} p_d^{u_d+1}$. The edges will be labeled with all integers I where $2 \le I \le k+1$. For odd i, construct $N-k$ edges between $\tilde{v}_i$ and $\tilde{v}_{i+1} \mod 2\prod_{d=1}^{t} p_d^{u_d+1}$. One of these edges will be labeled with 1 and the rest with all integers I where $k+2 \le I \le N$. As a result, for $i \ne 1, k+1$, between $\widetilde{v_i}$ and $\widetilde{v_{i+1}}$, there is a copy of a two-cycle that is a double cover of the nonreflection edges in $\mathscr{D}_\Gamma$ labeled with i and $(i+1) \mod N$. We will then have a graph with $\prod_{d=1}^{t} p_d^{u_d+1}$ copies of two-cycles labeled with i and $(i+1)$ for $2 \le i \le N, i \ne k+1$, as well as two $2\prod_{d=1}^{t} p_d^{u_d+1}$-cycles, one labeled with 1 and 2, and one labeled with $k+1$ and $k+2$. For an example, refer to the graph on the right in Figure 5, which is an 18-sheeted cover of the singular set from the orbicomplex in Figure 2.

We now consider the special case where $k = N$. For even i, construct $N-2$ edges between $\tilde{v}_i$ and $\tilde{v}_{i+1}$ labeled with integers I where $2 \le I \le N-1$. Then for odd i, construct 2 edges between vertices $\tilde{v}_i$ and $\tilde{v}_{i+1}$ labeled with 1 and N. We will again have a graph with $\prod_{d=1}^{t} p_d^{u_d+1}$ copies of two-cycles labeled with i and $(i+1)$ for $i = N$ and $2 \le i \le N-2$. We will also still have two $2\prod_{d=1}^{t} p_d^{u_d+1}$-cycles, one labeled with $N-1$ and N and the other with 1 and 2 as before.

Construction of $\widetilde{S_2}$: For the second singular set $\widetilde{S_2} \subset \widetilde{X_2}$, first partition $\{\tilde{v'}_i\}$ into $p = \prod_{d=1}^{t} p_d$ sets of equal size, so the first set will contain vertices $\tilde{v'}_i$ where $1 \le i \le 2\prod_{d=1}^{t} p_d^{u_d}$, the second set will contain vertices where $2\prod_{d=1}^{t} p_d^{u_d} + 1 \le i \le 4\prod_{d=1}^{t} p_d^{u_d}$, so in general, the nth set will contain $\tilde{v'}_i$ labeled $2n\left(\prod_{d=1}^{t} p_d^{u_d}\right) + 1 \le i \le 2(n+1)\prod_{d=1}^{t} p_d^{u_d}$, where $0 \le n \le p-1$.

We first consider the case where $k \ne N$. Construct $k-1$ edges between the pairs of vertices labeled $\tilde{v'}_{2n\left(\prod_{d=1}^{t} p_d^{u_d}\right)+1}$ and $\tilde{v'}_{2(n+1)\prod_{d=1}^{t} p_d^{u_d}}$, for $0 \le n \le p-1$. Label these edges with integers I, where $2 \le I \le k$. For all $1 \le n \le p$, construct an edge each labeled with $k+1$ between $\tilde{v'}_{2n\prod_{d=1}^{t} p_d^{u_d}}$ and $\tilde{v'}_{2n\left(\prod_{d=1}^{t} p_d^{u_d}\right)+1}$. Finally, add edges to the remaining vertices with no edges between them. For even i, if there are no edges between $\tilde{v'}_i$ and $\tilde{v'}_{i+1}$, add k edges and label them with integers I where $2 \le I \le k+1$. For odd i, add k edges between all $\tilde{v'}_i$ and $\tilde{v'}_{i+1}$ with no edges between them. One of these edges will be labeled with a 1, while the rest are labeled with integers I such that $k+2 \le I \le N$. As a result, we will have p cycles of length $2\prod_{d=1}^{t} p_d^{u_d}$ labeled with 1 and 2, $\prod_{d=1}^{t} p_d^{u_d+1}$ two-cycles labeled with i and $i+1$ for $2 \le i \le k$ or $k+2 \le i \le N$, $\prod_{d=1}^{t} p_d^{u_d}$ two-cycles labeled with k and $k+1$, and a $2p$-cycle labeled with k and $k+1$. For an example, see the graph on the left in Figure 5.

Now consider the edge case of $k = N$. This time, we will construct one edge labeled with 1 between the pairs of vertices labeled $\tilde{v'}_{2n\left(\prod_{d=1}^{t} p_d^{u_d}\right)+1}$ and $\tilde{v'}_{2(n+1)\prod_{d=1}^{t} p_d^{u_d}}$, for $0 \le n \le p-1$. For all $1 \le n \le p$, construct an edge labeled with N between $\tilde{v'}_{2n\prod_{d=1}^{t} p_d^{u_d}}$ and $\tilde{v'}_{2n\left(\prod_{d=1}^{t} p_d^{u_d}\right)+1}$. For the other pairs of vertices, for even i, if there are no edges between $\tilde{v'}_i$ and $\tilde{v'}_{i+1}$, add 2 edges and label them with 1 and N. For odd i, add $N-2$ edges between all $\tilde{v'}_i$ and $\tilde{v'}_{i+1}$ labeled with integers I such that $2 \le I \le N-1$. As a result, we will have p cycles of

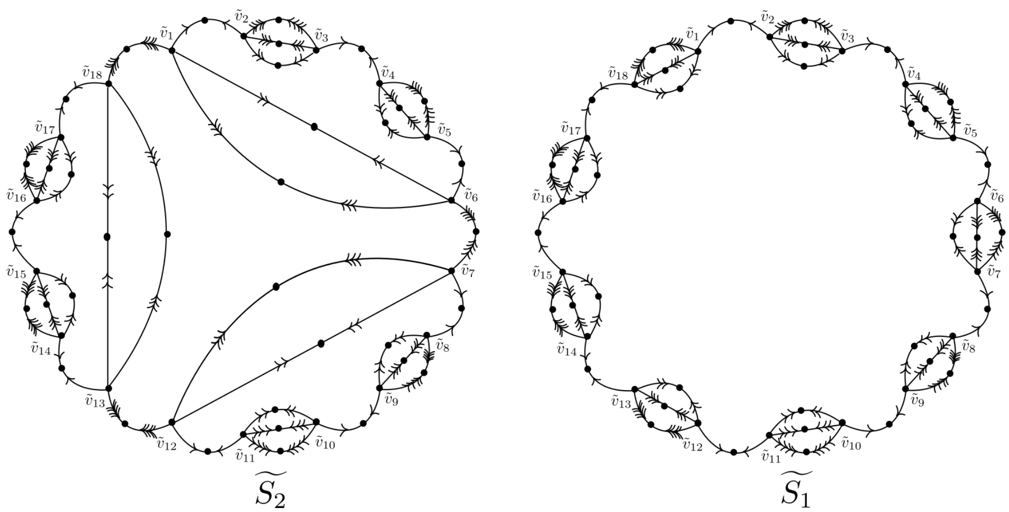

Figure 5: Two homotopic but non-homeomorphic covers of the singular set of W_Γ from Figure 2. Instead of labeling the edges with numbers, we use different numbers of arrows.

length $2\prod_{d=1}^{t} p_d^{u_d}$ labeled with 1 and 2, $\prod_{d=1}^{t} p_d^{u_d+1}$ each of two-cycles labeled with i and $i+1$ for $2 \leq i \leq N-2$ or $i = N$, and a $2p$-cycle labeled with 1 and N.

Observe that $\widetilde{S_1}$ and $\widetilde{S_2}$ are not homeomorphic since $\widetilde{S_1}$ has many more cut pairs. In particular, any two essential vertices will form a cut pair in $\widetilde{S_1}$, but $\widetilde{S_2}$ only has $\binom{2p}{2}$ pairs of cut pairs. As a result, $\widetilde{X}_1$ and $\widetilde{X}_2$ are not homeomorphic. However, note that $\widetilde{S_1}$ and $\widetilde{S_2}$ are homotopic; take a regular neighborhood of both graphs to obtain a $2\prod_{d=1}^{t} p_d^{u_d+1}(N-2)+1$-holed sphere.

We then examine what orbifolds are glued to $\widetilde{S_1}$. Recall that $r_{s,b}$ denotes the number of reflection edges of the orbifold corresponding to the bth branch of $\Theta_s \subset \Gamma$. By Lemma 2.1, we can see that there will be a disk with $\prod_{d=1}^{t} p_d^{u_d+1}(r_{s,b}-3)+2$ cone points glued along its boundary circle to the $2\prod_{d=1}^{t} p_d^{u_d}$-cycles labeled by 1 and 2 and $k+1$ and $k+2$ for $s = 1, k+1$ for the case $k \neq N$; if $k = N$, the second $2\prod_{d=1}^{t} p_d^{u_d}$-cycle is labeled with $N-1$ and N and $s = N-1$. Additionally, there will be a disk with $r_{i,b}-1$ cone points glued to each of the $\prod_{d=1}^{t} p_d^{u_d+1}$ two-cycles labeled by i and $i+1 \mod N+1$ for $2 \leq i \leq k$ or $k+2 \leq i \leq N$ if $k \neq N$ ($2 \leq i \leq N-2$ or $i = N$ if $k = N$).

Next, we list the orbifolds glued to $\widetilde{S_2}$. First, there will be p copies of disks with $\prod_{d=1}^{t} p_d^{u_d}(r_{1,b}-3)+2$ cone points glued to the $2\prod_{d=1}^{t} p_d^{u_d}$-cycles with edges labeled with 1 and 2. Second, there will be one copy of a disk with $\prod_{d=1}^{t} p_d^{u_d+1}(r_{k+1,b}-3)+2$ cone points glued to the $2\prod_{d=1}^{t} p_d^{u_d+1}$ cycle labeled by $k+1$ and $k+2$ if $k \neq N$ and $N-1$ and N if $k = N$. There will also be $\prod_{d=1}^{t} p_d^{u_d+1}$ copies of a disk with $r_{i,b}-1$ cone points glued to each 2-cycle labeled by i and $i+1 \mod N$ for $2 \leq i \leq k-1$ or $k+1 \leq i \leq N$ if $k \neq N$ and $2 \leq i \leq N-2$ if $k = N$. Finally, there are $\prod_{d=1}^{t} p_d^{u_d}$ copies of a disk with $r_{k,b}-1$ cone points glued to each two-cycle labeled by k and $k+1$ as well as one copy of a disk with $p(r_{k,b}-3)+2$ cone points glued to the $2p$-cycle labeled by k and $k+1$. Then for both orbicomplex covers, we have the same collection of orbifolds glued to $\widetilde{S_1}$ and $\widetilde{S_2}$:

- $\prod_{d=1}^{t} p_d^{u_d+1}$ copies of each disk with $r_{i,b}-1$ cone points, where $2 \le i \le k-1$ or $k+1 \le i \le N$ if $k \ne N$ and $2 \le i \le N-2$ if $k = N$;

- One copy of a disk with $\prod_{d=1}^{t} p_d^{u_d+1}(r_{k+1,b}-3)+2$ cone points;

- $\prod_{d=1}^{t} p_d^{u_d} + p = \prod_{d=1}^{t} p_d^{u_d+1}$ copies of a disk with $r_{k,b}-1 = \prod_{d=1}^{t} p_d^{u_d}(r_{1,b}-3)+2$ cone points since there are $\prod_{d=1}^{t} p_d^{u_d}$ copies of two-cycles labeled with k and $k+1$ as well as p copies of $2\prod_{d=1} p_d^{u_d}$-cycles labeled with 1 and 2 in $\widetilde{S_2}$ and $\prod_{d=1}^{t} p_d^{u_d+1}$ copies of two-cycles labeled with k and $k+1$ in $\widetilde{S_1}$;

- One copy of the disk with $p(r_{k,b}-3)+2 = \prod_{d=1}^{t} p_d^{u_d+1}(r_{1,b}-3)+2$ cone points since there is one $2p$-cycle labeled with k and $k+1$ in $\widetilde{S_2}$ and one $2\prod_{d=1}^{t} p_d^{u_d+1}$-cycle labeled with 1 and 2 in $\widetilde{S_1}$.

As a result, since $\widetilde{X_1}$ and $\widetilde{X_2}$ have the same sets of jester hats glued to homotopic graphs, we have found finite covers that are homotopic but not homeomorphic. This proves Theorem 1.7.

□

Next, we prove a Lemma which immediately implies Theorem 1.10.

Lemma 3.4. *Suppose Γ_1 and Γ_2 form a non-isomorphic permuted pair (see Definition 1.8). Then there exist finite-sheeted covers of D_{Γ_1} and D_{Γ_2} that are homotopic but not homeomorphic.*

Proof. Note that $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$ have the same number of generalized Θ graphs glued together, so the two-sheeted covers of their singular sets will be two isomorphic generalized Θ graphs S_1 and S_2. We will now construct two non-homeomorphic double covers of S_1 and S_2, which we will call $\widetilde{S_1}$ and $\widetilde{S_2}$.

Since Γ_1 and Γ_2 are not isomorphic, $N > 3$, where N as before denotes the number of generalized Θ graphs glued together in Γ_1 and Γ_2. Indeed, suppose that Γ_1 and Γ_2 are cycles of three generalized Θ graphs, Θ_1, Θ_2, and Θ_3. Without loss of generality, suppose that in Γ_1, Θ_i has essential vertices v_i and v_{i+1} and in Γ_2, Θ_1 has essential vertices v_1' and v_2', Θ_2 has essential vertices v_1' and v_3', and Θ_3 has essential vertices v_2' and v_3'. Then there is a graph isomorphism $f : \Gamma_1 \to \Gamma_2$, defined by $f(v_1) = v_2'$, $f(v_2) = v_1'$, and $f(v_3) = v_3'$ (with f defined on the valence 2 vertices in the natural way). We point this out since the construction detailed below will not work for $N = 3$.

To construct $\widetilde{S_1}$, fix a cyclic ordering on a set of four vertices $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\}$. Construct edges between $\tilde{v}_1$ and $\tilde{v}_2$, and between $\tilde{v}_3$ and $\tilde{v}_4$, labeled with all integers I such that $1 \le I \le N-1$. Then construct one edge between $\tilde{v}_2$ and $\tilde{v}_3$, as well as $\tilde{v}_4$ and $\tilde{v}_1$ and label those edges with N.

According to the assumptions, for every generalized Θ graph Θ_i between vertices v_i and v_{i+1} in Γ_1, there is an isomorphic generalized Θ graph Θ_j' in Γ_2 between v_j' and $v_{j+1}' \in \Gamma_2$. We can therefore assume there exists some $\Theta_j' \subset \Gamma_2$ that is isomorphic to $\Theta_1 \subset \Gamma_1$, and some $\Theta_k' \subset \Gamma_2$ that is isomorphic to $\Theta_N \subset \Gamma_1$. Without loss of generality, assume that $j < k$. Construct $\widetilde{S_2}$ with vertices $\{\tilde{v'}_1, \tilde{v'}_2, \tilde{v'}_3, \tilde{v'}_4\}$ and construct edges between $\tilde{v}_1$ and $\tilde{v'}_2$ as well as $\tilde{v'}_3$ and $\tilde{v'}_4$ labeled with all integers I such that $j+1 \le I \le k$. Then construct edges between $\tilde{v'}_2$ and $\tilde{v'}_3$ as well as $\tilde{v'}_4$ and $\tilde{v'}_1$ labeled with all integers I such that $k+1 \le I \le N$ or $1 \le I \le j$.

In the Davis orbicomplexes, the isomorphic Θ graphs $\Theta_1 \subset \Gamma_1$ and $\Theta_j' \subset \Gamma_2$ give rise to identical sets of orbifolds glued to edges labeled 1 and 2 in D_{Γ_1} and j and $j+1$ in D_{Γ_2}. As a result, if there is a cycle in X_1 labeled with 1 and 2 and a cycle of the same length in X_2 labeled with j and $j+1$, the sets of jester hats

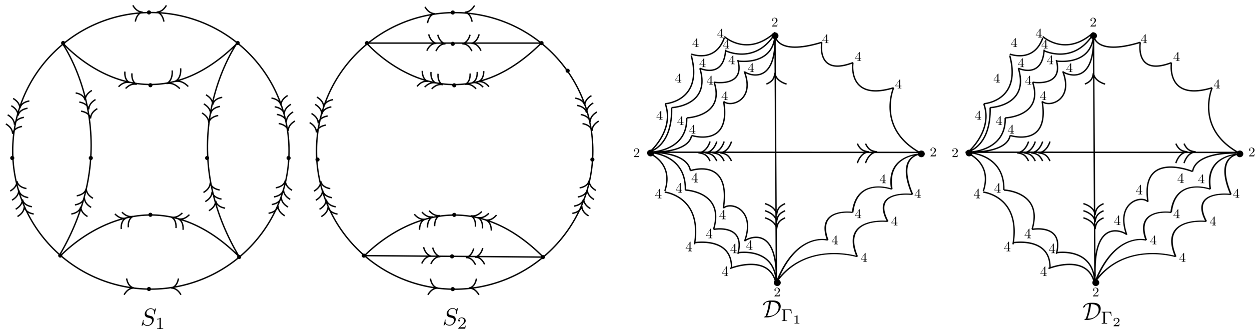

Figure 6: Here, the defining graphs Γ_1 and Γ_2 are the permuted pair from Figure 1, so they satisfy the conditions of Lemma 3.4. The singular sets S_i of X_i, which are four-sheeted covers of $\mathscr{D}_{\Gamma_i}$ for $i = 1,2$ are shown on the left. Note that X_1 and X_2 are homotopic but not homeomorphic.

glued to them will be identical. The same holds for $\Theta_N \subset \Gamma_1$ and $\Theta'_k \subset \Gamma_2$. Note that in $\widetilde{S}_1$, there are two four-cycles with jester hats glued to them, one labeled with 1 and N and the other with N and $N-1$. The other cycles with jester hats glued to them are all two-cycles labeled with I and $I+1$ where $1 \leq I \leq N-2$. On the other hand, $\widetilde{S}_2$ is a generalized Θ graph with two cycles of length four that have jester hats glued to them- one labeled with k and $k+1$ and the other with j and $j+1$. The other cycles with jester hats glued to them are labeled with I and $I+1$ where $I \neq j,k$ are two-cycles. Note that the regular neighborhood of both $\widetilde{S}_1$ and $\widetilde{S}_2$ is the $(2N-2)$-holed sphere $S_{0,2N-2}$, and there is a bijective correspondence between sets of jester hats glued to boundary components of $S_{0,2N-2} \subset \widetilde{S}_1$ and sets of jester hats glued to boundary components of $S_{0,2N-2} \subset \widetilde{S}_2$. Thus, X_1 and X_2 are homotopic but not homeomorphic because $\widetilde{S}_1$ and $\widetilde{S}_2$ are not homeomorphic. For an example, refer to Figure 6.

□

As a segue into our next section, we make the following remark.

Remark 3.5. Finding necessary and sufficient conditions for topological rigidity is a very nuanced task. In our setting, by Theorem 1.10, we know a topologically rigid set cannot contain the finite-sheeted covers of $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$, where Γ_1, Γ_2 are strongly repetitive or form a permuted pair (see Definition 1.8). Unfortunately, simply excluding finite-sheeted covers of all $\mathscr{D}_\Gamma$ where Γ is repetitive and part of a permuted pair is not sufficient for constructing a topologically rigid set. For example, for $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$ from Figure 7, Γ_1 and Γ_2 are not composed of the same set of generalized Θ graphs glued together. Furthermore, for both defining graphs, there do not exist pairs of commensurable Euler characteristic vectors of generalized Θ graphs in Γ_1 and Γ_2. Nevertheless, there exist eight- and four-sheeted covers of $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$ respectively that are homotopic but not homeomorphic.

4 A topologically rigid set

In this section, we introduce a class of finite-sheeted covers of Davis orbicomplexes that is topologically rigid. Since topological rigidity is difficult to achieve, many assumptions are necessary, so our class does not contain the complete set of finite-sheeted covers of Davis orbicomplexes. In particular, to find rigid classes, we not only need restrictions on the defining graphs but also on the singular sets of the covers.

A key tool in the proof of Theorem 4.7 is Lafont's topological rigidity result from [5]. Lafont's result involves (simple, thick, 2-dimensional) *hyperbolic P-manifolds* (see [5] Definition 2.3), which, roughly speaking, consist of compact surfaces with boundary identified along their boundaries. The gluing curves

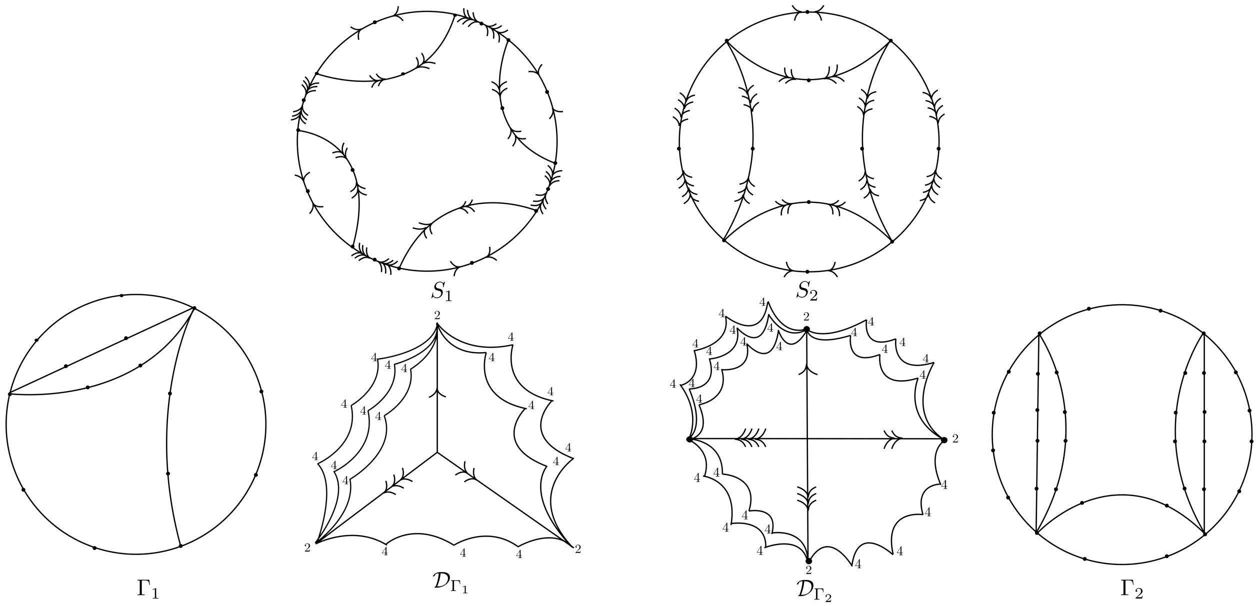

Figure 7: S_1 and S_2 are non-homeomorphic singular sets of homotopic eight- and four-sheeted covers of $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$. Thus, neither Γ_1 nor Γ_2 is repetitive and Γ_1 and Γ_2 do not form a permuted pair, yet any set $\mathscr{X}''$ that contains all finite-sheeted covers of $\mathscr{D}_{\Gamma_1}$ and $\mathscr{D}_{\Gamma_2}$ is not topologically rigid.

form the *singular set* of the hyperbolic P-manifold. Additionally, there is the important restriction that the singular set of a hyperbolic P-manifold consists of disjoint unions of circles. We now restate the theorem we will use:

Theorem 4.1 (Lafont [5], Theorem 1.2)**.** *Let X_1, X_2 be a pair of simple, thick, 2-dimensional hyperbolic P-manifolds, and assume that $\phi : \pi_1(X_1) \to \pi_1(X_2)$ is an isomorphism. Then there exists a homeomorphism $\phi : X_1 \to X_2$ that induces ϕ on the level of the fundamental groups.*

In order to use Theorem 4.1, we impose a restriction on the finite covers of Davis orbicomplexes we are examining, which we state below.

Assumption 4.2. *X is homotopic to an orbicomplex Y that consists of jester hats glued along their boundaries to the boundaries of an h-holed genus g surface $S_{g,h}$. Furthermore, in Y, each boundary of $S_{g,h}$ has at least one jester hat glued to it.*

Recall that a graph Γ is said to be *3-convex* if every edge between its essential vertices has at least 3 subdivisions. Note that if a cycle of generalized Θ graphs Γ is 3-convex, then W_Γ is hyperbolic since Γ is square-free. Although the converse is not true, we impose the 3-convexity condition in our proof to ensure our construction of hyperbolic P-manifold lifts of Davis orbicomplexes will work.

We now use Assumption 4.2 to prove a lemma that will be important in the proof of our main result of the section.

Lemma 4.3. *Let X_1 and X_2 be finite covers of Davis orbicomplexes D_{Γ_1} and D_{Γ_2}, where Γ_1 and Γ_2 are 3-convex and satisfy Assumption 4.2, and suppose $\pi_1(X_1) \cong \pi_1(X_2)$. Then the isomorphism $f : \pi_1(X_1) \to \pi_1(X_2)$ induces a bijection f_* between jester hats of X_1 and X_2 and for a jester hat $\mathscr{O}_1 \subset X_1$, if $f_*(\mathscr{O}_1) = \mathscr{O}_2 \subset X_2$, then $\mathscr{O}_1$ and $\mathscr{O}_2$ are homeomorphic. Furthermore, if S_1 and S_2 are singular subsets of X_1 and X_2 respectively, if $\gamma_1 \subset S_1$ is the boundary component of $\mathscr{O}_1$, then $f_*(\gamma_1) = \gamma_2$ where γ_2 is the boundary component of $\mathscr{O}_2$.*

Proof. Suppose X_1 and X_2 are two orbicomplexes with 3-convex defining graphs where $\pi_1(X_1) \cong \pi_1(X_2)$. We first use the construction from Proposition 3.2 of [7]: let X_i be a finite cover of $\mathscr{D}_{\Gamma_i}$ for $i = 1,2$. Each jester hat in X_i with p cone points lifts to a orbifold with $2(p-2)$ cone points and two boundary components, which in turn has a two-sheeted cover $S_{g,4}$, where $g = \frac{2(p-2)}{2} - 1$. Then, we glue each boundary component of $S_{g,4}$ to a copy of S_i, the singular set of X_i, to obtain a torsion-free four-sheeted cover. We will call these torsion-free covers $\widehat{X_1}$ and $\widehat{X_2}$. See Figure 8 for an illustration of the construction.

Since abstract commensurability is an equivalence relation, $\pi_1(\widehat{X_1})$ and $\pi_1(\widehat{X_2})$ are abstractly commensurable, so there exist finite-sheeted covers $\mathscr{Y}_1$ and $\mathscr{Y}_2$ such that

$$\pi_1(\widehat{X_1}) \geq \pi_1(\mathscr{Y}_1) \cong \pi_1(\mathscr{Y}_2) \leq \pi_1(\widehat{X_2}).$$

Note that $\widehat{X_1}$ and $\widehat{X_2}$ are homotopic to hyperbolic P-manifolds; take the regular neighborhoods of their singular sets. As a result, finite-sheeted covers of $\widehat{X_1}$ and $\widehat{X_2}$ are also homotopic to hyperbolic P-manifolds by Nielsen-Schreier, so it follows that $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are homotopic to hyperbolic P-manifolds Y_1 and Y_2. By Corollary 3.5 of [5], a homotopy ϕ between hyperbolic P-manifolds induces a bijection between homeomorphic chambers (hyperbolic manifolds with boundary) of Y_1 and Y_2. Additionally, for a chamber $C_1 \subset Y_1$ with boundary component γ_1, if $\phi(C_1) = C_2 \subset Y_2$, then $\phi(\gamma_1) = \gamma_2$ where γ_i is a boundary component of C_i for $i = 1,2$. Using these results, we can conclude that surfaces with boundary in $\mathscr{Y}_1$ are mapped bijectively, and homeomorphically, to surfaces in $\mathscr{Y}_2$, as the maps are preserved under homotopy. The homotopy lifting property then gives us the statement of the lemma.

Alternatively, observe that the Davis complex Σ_Γ, the universal cover of $\mathscr{D}_\Gamma$, is CAT(0) and thus contractible. Since W_Γ acts freely on Σ_Γ, it follows that $\mathscr{D}_\Gamma$ is a classifying space of W_Γ, or equivalently a $K(W_\Gamma, 1)$ space. Furthermore, finite covers of $\mathscr{D}_\Gamma$ are quotients of Σ_Γ by a free action as well, so X_1 and X_2 are classifying spaces for the same finite-index subgroup of W_Γ. As a result, since $\pi_1(X_1) \cong \pi_1(X_2)$, X_1 and X_2 are homotopy equivalent by Whitehead's Theorem. This allows us to construct a shorter tower of covers since a homotopy between X_1 and X_2 induces a homotopy between $\widehat{X_1}$ and $\widehat{X_2}$, which are homotopic to hyperbolic P-manifolds. We can then apply Theorem 4.1 to obtain our result. □ □

We stress that Assumption 4.2 is key for the proof of Lemma 4.3. For example, let $S_{g,n}$ be a genus g surface with n boundary components. Recall the *graph genus* of a graph G is the minimal genus of an orientable surface into which G can be embedded. In general, a cover of a Davis orbicomplex X is homotopic to an orbicomplex consisting of jester hats identified along their boundaries to a set of simple closed curves C on $S_{g,n}$ since every graph has a genus (see [8]). Analyzing X can be difficult since there is no guarantee of the four-sheeted hyperbolic P-manifold cover constructed in Lemma 4.3, as the lifts of C may not be a disjoint union of circles. For example, consider Figure 9, which depicts a six-sheeted cover of a Davis orbicomplex $\mathscr{D}_\Gamma$. The singular set of X is the complete bipartite graph $K_{3,3}$, which is homotopic to $S_{1,3}$. Since the jester hats are not identified along disjoint circles, Lafont's rigidity result is not available for use and the proof for Lemma 4.3 does not work.

In order to determine whether two finite-sheeted covers of Davis orbicomplexes are homeomorphic, which we need to do to determine topological rigidity, we need to check that their singular sets are homeomorphic. Unfortunately, since finite covers of the singular sets are graphs, determining topological rigidity therefore requires solving a graph isomorphism problem, which has a high computational complexity. Recall that a *complete graph invariant* is combinatorial tool for determining whether a pair of graphs in a family of graphs is isomorphic. To simplify our problem, we will define a family of singular sets $\mathscr{S}$ of finite-sheeted covers of a Davis complex $\mathscr{D}_\Gamma$ with an easily computable complete graph invariant, which we now introduce.

Definition 4.4 (Cycle count vectors)**.** Consider X, a $2d$-sheeted cover of a Davis orbicomplex, where $d > 0$ is any arbitrary integer. Recall that for a cycle of generalized Θ graphs Γ, its associated Davis orbicomplex

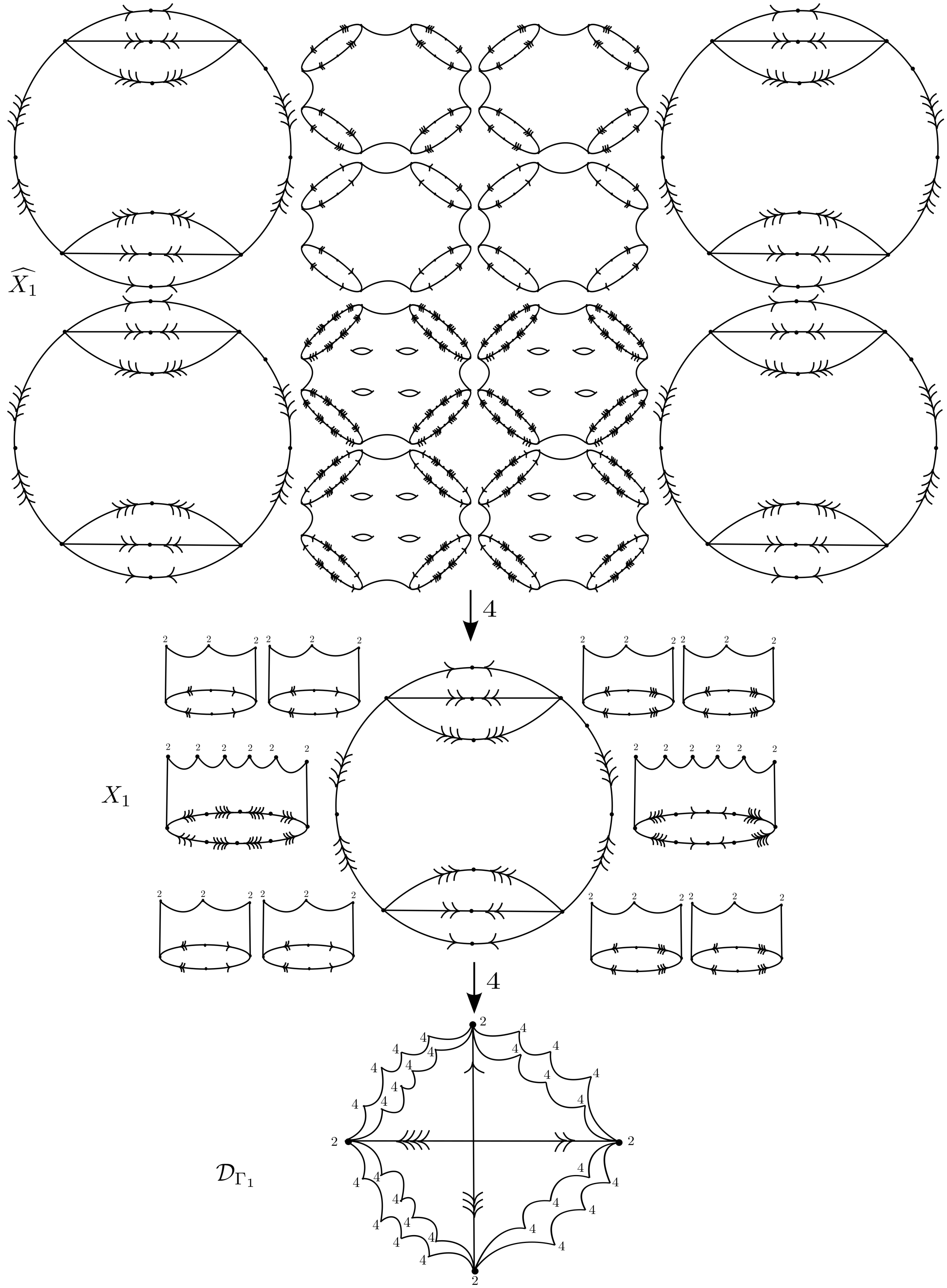

Figure 8: A tower of covers constructed in the proof of Lemma 4.3. Note that $\widehat{X_1}$ is homotopic to a hyperbolic P-manifold.

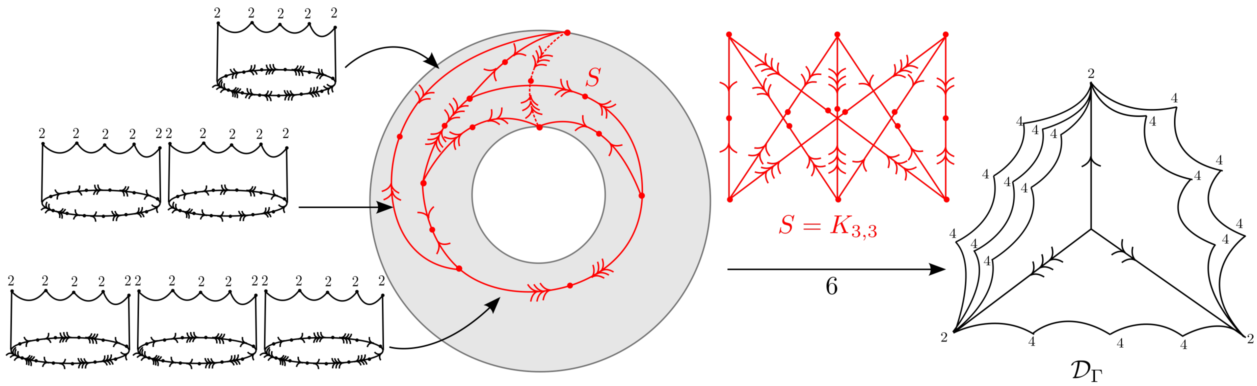

Figure 9: An example of a finite-sheeted cover of a Davis orbicomplex $\mathscr{D}_\Gamma$ that does not satisfy Assumption 4.2. The singular set S is not necessarily planar; in this example, S (depicted in red) embeds on a torus.

$\mathscr{D}_\Gamma$ has a singular set that is a star graph with N edges, which we will label with integers $i = 1,2,\ldots,N$. If an edge $e = [v,w]$ is labeled by i, we will write $e = [v,w]_i$. Fix a cyclic labeling of the edges, which will lift to a labeling in any cover of the singular set. For $1 \le i \le N$, let $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,d})$ be a vector where $x_{i,j}$ is the number of cycles in S of length $2j$ that are labeled with i and $i+1$. Recall that as usual, we are counting the edges between two essential vertices as one edge. Note that x_i is a vector of length d since the possible cycle lengths of a $2d$-sheeted cover will range from 2 to $2d$. For an example, refer to Figure 6. Let the labeling of an edge of S_i be the number of arrows seen on the edge, so $N = 3$. For S_1, the set of cycle count vectors is $x_1 = (2,0)$, $x_2 = (0,1)$, $x_3 = (2,0)$, and $x_4 = (0,1)$ since there are two 2-cycles labeled with 1 and 2, two 2-cycles labeled with 3 and 4, one 4-cycle labeled with 2 and 3, and one 4-cycle labeled with 4 and 1. Similarly, for S_2, the set of cycle count vectors is $x_1 = (2,0)$, $x_2 = (2,0)$, $x_3 = (0,1)$ and $x_4 = (0,1)$. As we will see soon, since the cycle count vectors are different, S_1 and S_2 are not isomorphic.

We now define a family of singular sets of finite-sheeted covers of a Davis complex $\mathscr{D}_\Gamma$. Recall that the double cover of the singular set of $\mathscr{D}_\Gamma$ is itself a generalized Θ graph Θ_N with N branches, where N is the number of generalized Θ graphs in the defining graph Γ (see Construction 3.2). A double cover of Θ_N is a cycle of four (possibly trivial) generalized Θ graphs S' with valence N vertices. Note there exists some $a,b \in \mathbb{Z}_{\ge 0}$ and $a+b = N$ such that adjacent essential vertices of S' either have a or b branches between them. For example, in Figure 6, all the essential vertices in S_1 and S_2 have valence $N = 4$ since the original defining graph consisted of four generalized Θ graphs glued together. In S_1, $a = 2$ and $b = 2$, and in S_2, $a = 1$ and $b = 3$.

Construction 4.5 (A special class of singular sets $\mathscr{S}$)**.** To begin our construction, take S', a four-sheeted cover of the singular set of $\mathscr{D}_\Gamma$, which is a cycle of four generalized Θ graphs each with either a or b branches. Then arbitrarily choose two adjacent vertices of valence greater than two, v_i and v_{i+1} with $n_i = a$ or b edges between them. Delete some fixed number of (subdivided) edges j between them, where $1 \le j \le n_i$, add two essential vertices u_i and u_{i+1}, and add j edges between v_i and u_i as well as v_{i+1} and u_{i+1}. Finally, add $N - j$ edges between u_i and u_{i+1}. Then arbitrarily choose two other adjacent essential vertices and repeat the process any finite number of times. See Figure 10 for an example of an element of $\mathscr{S}$; at each step, two edges are added between the new essential vertices.

Let $S \in \mathscr{S}$ be a graph that can be constructed from the process described above. By construction, S covers the original singular set of the Davis orbicomplex. Notice $\mathscr{S}$ describes singular sets, not defining graphs, so the graphs in $\mathscr{S}$ are not restricted to cycles of generalized Θ graphs. We now introduce a second assumption:

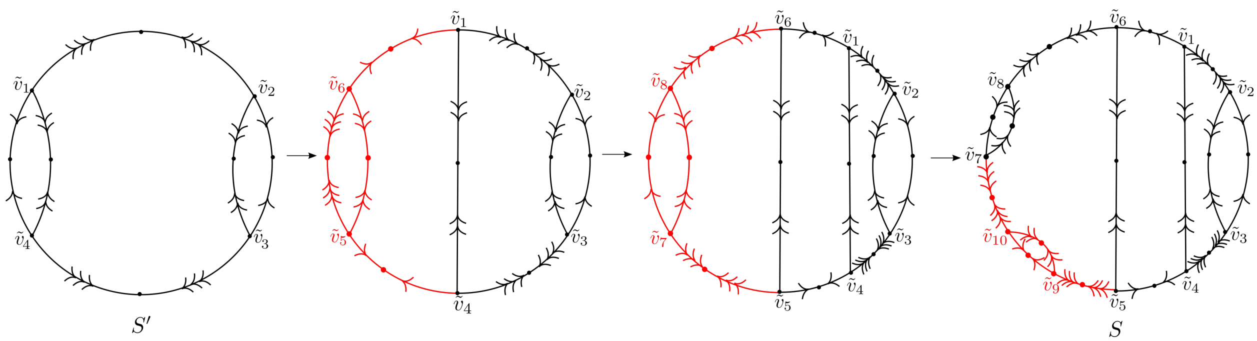

Figure 10: Construction of a $S \in \mathscr{S}$ from a cycle of generalized Θ graphs. The edges and vertices added at each step are depicted in red.

Assumption 4.6. The singular set S of X is an element of the set $\mathscr{S}$ defined in Construction 4.5.

We are now ready to introduce a topologically rigid class finite covers of Davis orbicomplexes.

Theorem 4.7. *Suppose Γ is 3-convex and not repetitive (see Definition 1.6). Let $\mathscr{X}''$ contain all finite-sheeted covers of $\mathscr{D}_\Gamma$ that satisfy Assumptions 4.2 and 4.6. Then $\mathscr{X}''$ is topologically rigid.*

We now give an outline of the proof of Theorem 4.7 (for the actual proof, see the end of the section). We first show that the statement of the theorem reduces to a graph isomorphism problem on singular sets of two finite covers of Davis orbicomplexes (see Lemma 4.10). We then show that if two finite covers of Davis orbicomplexes satisfying the conditions listed in Theorem 4.7 are homotopic, then they have the same cycle count vectors. Finally, in Lemma 4.12, we show that cycle count vectors are a complete graph invariant for singular sets in $\mathscr{S}$ detailed in Construction 4.5.

Remark 4.8. Recall that if X_1 and X_2 are homeomorphic, then W_{Γ_1} and W_{Γ_2} are commensurable. Given that Assumption 4.2 is true, the converse is very much false. Figure 1.2 of [2] gives some examples of pairs of defining graphs $\{\Gamma_i, \Gamma_i'\}$ $(i = 1,2,3)$ of commensurable RACGs, which we will now reference. We can check that none of the finite-sheeted covers of $\mathscr{D}_{\Gamma_i}$ and $\mathscr{D}_{\Gamma_i'}$ can be homotopic by Lemma 4.3, even though the RACGs W_{Γ_i} and $W_{\Gamma_i'}$ are commensurable. To see the full commensurability classification for cycles of generalized Θ graphs, refer to Theorem 1.12 of [2].

Definition 4.9. We say that a graph homeomorphism $\bar{f} : G_1 \to G_2$ is *label-preserving* if for all $[v,w]_i \in G_1$, $\bar{f}([v,w]_i) \subset G_2$ is also labeled by i.

We now prove the following useful result involving singular sets of finite covers of Davis orbicomplexes.

Lemma 4.10. *Let X_1 and X_2 satisfy the assumptions from Lemma 4.3. Then every label-preserving graph homeomorphism $\bar{f} : S_1 \to S_2$ between the singular subsets of X_1 and X_2 will induce a homeomorphism $f : X_1 \to X_2$.*

Proof. Suppose the singular sets $S_1 \subset X_1$ and $S_2 \subset X_2$ are homeomorphic. Then the vertices $v_j \in V(S_1)$ with valence greater than two map bijectively to the vertices $v_j' \in V(S_2)$ with the same valence, and if $\bar{f}(v_1) = v_1'$ and $\bar{f}(v_2) = v_2'$, then for every edge $[v_1, v_2]_i$ labeled with i, $\bar{f}([v_1,v_2]_i) = [v_1', v_2']_i$. As a result, every cycle $\gamma_1 \subset S$ labeled with i and $i+1 \pmod N$ is bijectively mapped to a cycle of the same length in $\gamma_2 \subset S_2$ labeled with i and $i+1 \pmod N$). By Lemma 4.3, for every jester hat $\mathscr{O}_1$ with c cone points glued to γ_1, there must also be a jester hat $\mathscr{O}_2$ with c cone points glued to $\gamma_2 = \bar{f}(\gamma_1)$. As a result, $\bar{f}$ induces a homeomorphism $f : X_1 \to X_2$ where $f(S_1) = S_2$ and $f(\mathscr{O}_1) = \mathscr{O}_2$. □ □

Recall that a graph G with genus g can be embedded into a genus g surface S_g. The edges of the graph will divide S_g into regions called *faces*. Let $|V|$ denote the number of vertices of G, $|E|$ the number of edges, and $|F|$ the number of faces. For planar graphs, we can calculate the number of faces using Euler's formula, $2 = |V| - |E| + |F|$. In general, using the definition of Euler characteristic for simplicial complexes, for a graph with genus g, $2-2g = |V| - |E| + |F|$.

Lemma 4.11. *Let $\mathscr{X}''$ be the set of finite-sheeted covers of a single Davis orbicomplex $\mathscr{D}_\Gamma$, where Γ is 3-convex and not repetitive. Suppose $X_1, X_2 \in \mathscr{X}''$ are homotopic, and their singular sets S_1 and S_2 satisfy the conditions listed in Theorem 4.7. Then $x_i = x'_i$ for all $1 \leq i \leq N$, where x_i and x'_i are the cycle count vectors of X_1 and X_2 respectively.*

Proof. First, we show that if Γ consists of N essential vertices, and X_i are d_i sheeted covers of $\mathscr{D}_\Gamma$ for $i = 1,2$, then $d_1 = d_2$ necessarily. As usual, we will denote S_i to be the singular set of X_i. Since X_1 and X_2 are homotopic, by Lemma 4.3, they consist of the same sets of jester hats identified along their boundary components to some F-holed genus g surface $S_{g,F}$. Note that F is also the number of faces of both $S_1 \subset X_1$ and $S_2 \subset X_2$. Note the number of vertices in S_i is d_i and the number of edges is $\frac{d_i N}{2}$ for $i = 1,2$, so using the definition of Euler characteristic, we have:

$$d_1 - \frac{d_1 N}{2} + F = 2 - 2g = d_2 - \frac{d_2 N}{2} + F \implies d_1\big(2-N\big) = d_2\big(2-N\big).$$

Thus, $d_1 = d_2$ necessarily. Then X_1 and X_2 are finite sheeted covers of the same degree of the same Davis orbicomplex $\mathscr{D}_\Gamma$.

Let $\Theta = \Theta(n_1, n_2, ..., n_k)$ and $\Theta' = \Theta(n'_1, n'_2, ..., n'_{k'})$ be two arbitrary Θ graphs in Γ. Recall that by Lemma 3.1, the number of cone points c_i of a jester hat corresponding to b_i, the ith branch of Θ, is $\frac{d}{2}(r_i - 3) + 2$ where d is the index of the cover the jester hat corresponds to. Similarly, the number of cone points c'_j of a jester hat corresponding to b'_j, the jth branch of Θ', is $\frac{d}{2}(r'_j - 3) + 2$. Note that Γ is not repetitive, so there do not exist K, L such that $K(\frac{1-n_1}{4}, \frac{1-n_2}{4}, ..., \frac{1-n_k}{4}) = L(\frac{1-n'_1}{4}, \frac{1-n'_2}{4}, ..., \frac{1-n'_{k'}}{4})$. So in particular, $(n_1 - 1, n_2 - 1, ..., n_k - 1) \neq (n'_1 - 1, n'_2 - 1, ..., n'_{k'} - 1)$ and since n_i and n'_j are equal to $r_i - 2$ and $r'_j - 2$ respectively, it follows that $\{\frac{d}{2}(r_i - 3) + 2\}_{1 \leq i \leq k} \neq \{\frac{d}{2}(r'_j - 3) + 2\}_{1 \leq j \leq k'}$. Thus, in any finite-sheeted cover of D_Γ, there are different sets of jester hats glued to cycles with different labels since the sets of cone point counts are different. In particular, jester hats in X_1 that are lifts of orbifolds corresponding to a Θ graph in Γ must map to a collection of jester hats in X_2 that are lifts of orbifolds corresponding to the same Θ graph. In order for the sets of jester hats to be the same, cycles of length $2j$ labelled with i and $i+1$ must map to cycles of the same length and also labeled with i and $i+1$. As a result, $x_i = x'_i$.

□ □

Lemma 4.12. *Let $\mathscr{X}''$ be the class of finite-sheeted covers of Davis orbicomplexes from Theorem 4.7. Suppose $S_1, S_2 \in \mathscr{S}$ are the singular sets of two $2d$- sheeted covers $X_1, X_2 \in \mathscr{X}''$. If $x_i = x'_i$ for all $1 \leq i \leq N$, then there exists a label-preserving homeomorphism $\bar{f} : S_1 \to S_2$.*

Proof. We use induction on the degree of the covers, $2d$. Note that since the cycle count vectors are the same, the covers must be of the same degree. For the base case, suppose $d = 2$, so $2d = 4$. By Assumption 4.2, each cycle in both S_1 and S_2 must be attached to at least one jester hat. Thus, cycles of odd length are not allowed since a jester hat cannot be attached to such a cycle. Thus, the only possible cycle lengths of S_1 and S_2 are 2 and 4. By construction, every edge in S_1 must be attached to part of the boundary of at least one jester hat, so every edge in S_1 must be included into at least one cycle. In order for S_1 to be a connected graph with four vertices satisfying the property that every edge is part of a at least one cycle, there must be at least one four-cycle in S_1. The same holds for S_2. Since three-cycles are not allowed and each edge in S_1 will be included in a cycle, the only possible edges in S_1 are of the form $[v_i, v_{i+1}]$, where as before, i is taken modulo 4. The same holds for S_2. Therefore, the only possible S_1 and S_2 are graphs with k edges between v_1 and v_2 as well as v_3 and v_4 (where $1 \leq k < N$), and $N-k$ edges between v_2 and v_3 as well as v_4 and v_1 (namely, a cycle of 4 subdivided Θ graphs- see S_1 and S_2 in Figure 7 for examples of such graphs). Note that such graphs will have two four-cycles and $(2N-4)$ two-cycles. As a result, the only 4-sheeted covers that satisfy Assumption 4.2 of Theorem 4.7 consist of two sets of jester hats glued to four-cycles, and the rest of the sets of jester hats glued to two-cycles.

If $x_i = x_i'$ for all $1 \leq i \leq N$, we know that for both S_1 and S_2, there is one four-cycle labeled with j and $j+1$ for some $1 \leq j \leq N$, and another four-cycle labeled with k and $k+1$ for $k \neq j$. Without loss of generality, suppose $j < k$. Note that in S_1, if an edge labeled $j+1$ is between v_i and v_{i+1}, then an edge labeled k must necessarily also be between v_i and v_{i+1}. Otherwise, the edge labeled with $k+1$ must be between v_i and v_{i+1}, so an edge labeled with k is between v_i and v_{i-1}. In this case, the labels on the edges between v_i and v_{i+1} will range from $j+1$ to $k+1$, so one of the edges must be labeled with k. However, there is already a k edge between v_i and v_{i-1}, which is impossible. The same argument can be used for S_2 if we replace v_i with v_i'. We can thus see that in S_1, for all $1 \leq i \leq N$, if edges labeled with all integers between $j+1$ and k connect v_i and v_{i+1} (v_i' and v_{i+1}' in S_2), then the edges connecting v_i and v_{i-1} (v_i' and v_{i-1}' in S_2) are labeled with integers $1 \leq l \leq N$ such that $l \leq j$ or $l \geq k+1$. We can then construct a homeomorphism $\bar{f}: S_1 \to S_2$ where $f(v_i) = v_i'$ for all $1 \leq i \leq N$. We can easily check that edge labels and vertex adjacencies are preserved under $\bar{f}$, completing the base case.

Suppose the lemma holds for d-sheeted covers. By Assumption 4.6, there exist $\{v_i, v_{i+1}\} \in V(S_1)$ and $\{v_i', v_{i+1}'\} \in V(S_2)$ with the same number of edges and the same set of labels between them. Additionally, the only other edges attached to v_i and $v_{i+1} \in V(S_1)$ are also attached to v_{i-1} and v_{i+2} respectively; the same holds for $v_i', v_{i+1}' \in V(S_2)$. Note that for arbitrary $d > 0$, S_1 and S_2 must both have $(2d+2)$-cycles for the same reason the four-sheeted covers in the base case necessarily have 4-cycles: each edge of S_1 and S_2 is necessarily attached to a jester hat, and thus by Assumption 4.2 necessarily belongs to a cycle. Suppose there are no cycles in S_1 and S_2 of length $2d+2$. Then S_1 and S_2 would be disconnected since they are graphs with $2d+2$ vertices. Thus, there must be at least one $(2d+2)$-cycle in S_1 and S_2, which we will label with m and $m+1$. Note that if v_i and v_{i+1} have j edges between them, then the other two pairs of vertices $\{v_i, v_{i-1}\}$ and $\{v_{i+1}, v_{i+2}\}$ must also necessarily have $N-j$ edges between them, and the edges between the two pairs of vertices have the same set of labels. Delete v_i and v_{i+1} and the edges they are adjacent to, and construct the j deleted edges between v_{i+2} and v_{i-1} to create the singular set T_1 of a $2d$-sheeted cover of $\mathscr{D}_{\Gamma_1}$. Construct a singular set T_2 of $2d$-sheeted cover of $\mathscr{D}_{\Gamma_2}$ in the same way. Let y_i and y_i' be the new set of cycle count vectors. Note that in total, for both $2d$-sheeted covers, we have deleted and added the same set of cycles with the same set of edge labels, so $y_i = y_i'$ for all $1 \leq i \leq N$. Then by the inductive hypothesis, there exists a homeomorphism $\bar{g}: T_1 \to T_2$.

We then can extend $\bar{g}$ to $\bar{f}: S_1 \to S_2$. Suppose $\bar{g}(v_{i+2}) = v_k'$ for some $v_k' \in T_2$ (note that k is not necessarily equal to i). Then $\bar{g}(v_{i-1})$ maps to an adjacent vertex v_l such that there are j edges between v_k' and v_l' with the same labelings as the edges between v_i and v_{i+1} in the original S_1. Construct two vertices u_k and u_l in T_2 and v_{i-1} and v_{i+2} in T_1, and delete the $N-j$ edges between $v_k, v_l \in V(T_2)$ and $v_{i-1}, v_{i+2} \in V(T_1)$ that have the same labels as the edges added to construct T_i from S_i for $i = 1, 2$. Then reconstruct the j deleted edges between $u_k, u_l \in V(T_2)$ and $v_{i-1}, v_{i+2} \in V(T_1)$ as well as $N-j$ edges $[v_k', u_k]$ and $[v_l', u_l]$ in T_2 and $[v_{i-1}, v_i]$ and $[v_{i+1}, v_{i+2}]$ in T_1. Call the new graphs U_1 and U_2, but note that U_1 is identical to S_1. Additionally, U_2 is homeomorphic to S_2 since they are the same graph up to a relabeling of vertices. Let $\bar{g}(v) = \bar{f}(v)$ for all $v \in V(S_1)$, $\bar{f}(v_i) = u_k$ and $\bar{f}(v_{i-1}) = u_l$, which also determines the maps between the newly added edges, giving us a label-preserving homeomorphism $\bar{f}: S_1 \to S_2$. □ □

We now have all the tools to prove Theorem 4.7.

Theorem 4.7. It suffices to show that for $X_1, X_2 \in \mathscr{X}$, $\pi_1(X_1) \cong \pi_1(X_2)$ implies $X_1 \overset{\text{homeo}}{\cong} X_2$. As a result of

Lemma 4.10, in order to show $X_1 \overset{\text{homeo}}{\cong} X_2$, it suffices to show there exists a label-preserving graph homeomorphism between the singular sets of X_1 and X_2 respectively. Since subdivisions of a graph belong to the same homeomorphism class, we can delete the valence two vertices from the singular sets to obtain S_1 and S_2, and compare the graphs. By Lemma 4.11, $x_i = x'_i$ for $1 \leq i \leq N$, where x_i and x'_i are the cycle count vectors of S_1 and S_2 defined in Definition 4.4. Then by Lemma 4.12, we can conclude $S_1 \overset{\text{homeo}}{\cong} S_2$, as desired. □ □

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Email: yandi.wu@wisc.edu
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON