Iso-length-spectral Hyperbolic Surface Amalgams

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Abstract

Two negatively curved metric spaces are *iso-length-spectral* if they have the same multisets of lengths of closed geodesics. A well-known paper by Sunada provides a systematic way of constructing iso-length-spectral surfaces that are not isometric. In this paper, we construct examples of iso-length-spectral *surface amalgams* that are not isometric, generalizing Buser's combinatorial construction of Sunada's surfaces. We find both homeomorphic and non-homeomorphic pairs. Finally, we construct a noncommensurable pair with the same *weak length spectrum*, the length set without multiplicity.

1 Introduction

Recall that the *(unmarked) length spectrum* of a metric space (X, g) is the ascending multiset of positive real numbers representing lengths of closed geodesics in (X, g). We say that two metric spaces are *iso-length-spectral* if they have the same length spectra.

The question of whether there exist iso-length-spectral manifolds that are not isometric has a long history. The first examples of such objects are due to Milnor (see [Mil64]), who constructed two flat, 16-dimensional non-isometric, iso-length-spectral tori. In [Vig80], Vignerás constructed examples of hyperbolic surfaces that are iso-length-spectral but not isometric. Both Vignerás's and Milnor's constructions are heavily number theoretic and, one could argue, difficult to describe geometrically. In [Sun85], Sunada developed a celebrated, systematic, and more geometric way of constructing hyperbolic iso-length-spectral surfaces. In [Bus86] (and [Bus92] for the genus 6 case), Buser uses [Sun85] to construct pairs of iso-length-spectral, non-isometric surfaces of genus ≥ 5.

Due to the combinatorial nature of Buser's approach, one can apply his construction to metric spaces outside the setting of Riemannian manifolds. We explore such objects in this paper, and show that in contrast to the hyperbolic surface case, Buser's interpretation of the Sunada construction sometimes yields examples of *non-homeomorphic* iso-length-spectral objects.

Let (X, g) be a simple, thick hyperbolic surface amalgam, which, roughly speaking, is constructed by isometrically gluing together compact, hyperbolic surfaces with boundary together along their boundary components. We refer the reader to Definition 2.3 of [Laf07] for a more precise definition; note that in his paper, he refers to surface amalgams as "2-dimensional P-manifolds". We now state our first result:

Theorem 1.1. *There exist iso-length-spectral surface amalgams equipped with piecewise hyperbolic metrics that are* not *isometric. Furthermore, there exist non-homeomorphic iso-length-spectral pairs of hyperbolic surface amalgams.*

We remark that in contrast to the surface amalgam case, there cannot exist non-homeomorphic iso-length-spectral pairs of hyperbolic closed surfaces, as the length spectrum completely determines the genus of a surface (see [Cha84]). In the setting of 3-manifolds, on the other hand, applications of the Sunada method (e.g. [Spa89], [Rei92], [McR14]) have been shown to yield hyperbolic, iso-length-spectral 3-manifolds which are non-isometric and thus non-homeomorphic due to Mostow Rigidity.

This provides further evidence towards the fact that surface amalgams share characteristics of both surfaces and 3-manifolds (see also [HST20]).

We say that two metric spaces are *metrically commensurable* if they share some isometric finite-sheeted cover. Notably, both main sources of iso-length-spectral, non-isometric hyperbolic manifolds (from [Vig80] and [Sun85]) yield metrically commensurable manifolds. This led Reid to pose an interesting open question in [Rei92]:

Question 1.2 (Reid)**.** *Do there exist two iso-length-spectral hyperbolic manifolds which are not metrically commensurable?*

It is known that all compact iso-length-spectral *arithmetic* surfaces and 3-manifolds are necessarily metrically commensurable (see [Rei92] and [CHLR08]). In [LSV06], in contrast, the authors find large families of locally symmetric, iso-length-spectral manifolds of higher rank that are not metrically commensurable. However, the question of whether there exist iso-length-spectral, metrically non-commensurable hyperbolic surfaces is still open.

We say two topological spaces are *topologically commensurable* if they share *homeomorphic* (as opposed to isometric) finite-sheeted covers. Note that if two metric spaces are not topologically commensurable, they are automatically not metrically commensurable. For the remainder of the paper, when we say "commensurable," we mean topologically commensurable.

We follow the terminology from [PR15] and define the *weak length spectrum* of a locally CAT(-1) metric space to be a collection of lengths of closed geodesics *without* multiplicity. If two metric spaces have the same weak length spectrum, we say they are *weak length isospectral.* Notice that since the weak length spectrum is a subset of the length spectrum, weak length isospectrality is a weaker condition than length isospectrality.

In fact, in [LMNR07], the authors show that examples of weak length isospectral (called "length equivalent" in their paper) hyperbolic manifolds that are not iso-length-spectral exist in great abundance. The examples they construct arise from sequences of manifolds in towers of covers so that the weak length isospectral pairs have different volumes. In fact, the ratios between volumes of consecutive terms M_{n+1} and M_n in the sequences tend to infinity.

We are now ready to state the second main result of the paper:

Theorem 1.3. *There exist weak length isospectral surface amalgams equipped with piecewise hyperbolic metrics that are* not *(topologically) commensurable.*

We remark that it is impossible to find pairs of surfaces that are not topologically commensurable, as all closed surfaces are commensurable as topological objects. The proof of Theorem 1.3 relies on work from [Sta17] and [DST18] which is related to the abstract commensurability classification problem of right-angled Coxeter groups. Furthermore, in contrast to the examples from [LMNR07], the weak length isospectral pairs from Theorem 1.3 have the same volume.

Outline of the paper. We now give a brief outline of the paper. We begin with a brief review of Buser's techniques for constructing iso-length-spectral, non-isometric surfaces in Section 2. We continue with a construction of pairs of homeomorphic, iso-length-spectral, non-isometric surface amalgams, followed by a construction of a pair that is not homeomorphic using Lafont's criteria from [Laf07] in Section 3. Instead of using Buser's transplantation technique, we count copies of "identical" closed geodesics. Finally, in Section 4, we construct a pair of weak length isospectral surfaces which are not commensurable using criteria from [DST18]. This time, we prove weak length isospectrality using Buser's transplantation technique.

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2 Buser's Techniques

We rely heavily on the techniques from [Bus86], which we briefly sketch.

2.1 Buser's original construction

Buser constructs two iso-length-spectral but non-isometric hyperbolic genus 5 surfaces by gluing together identical copies of right-angled octagons, which he calls *building blocks*. The gluing scheme is shown below in Figure 1. His construction is modeled off Sunada's construction using almost conjugate subgroups of $(\mathbb{Z}/8\mathbb{Z})^\times \ltimes (\mathbb{Z}/8\mathbb{Z})^+$ given by Gerst in [Ger70] (see Example 1 of Section 1 of [Sun85]).

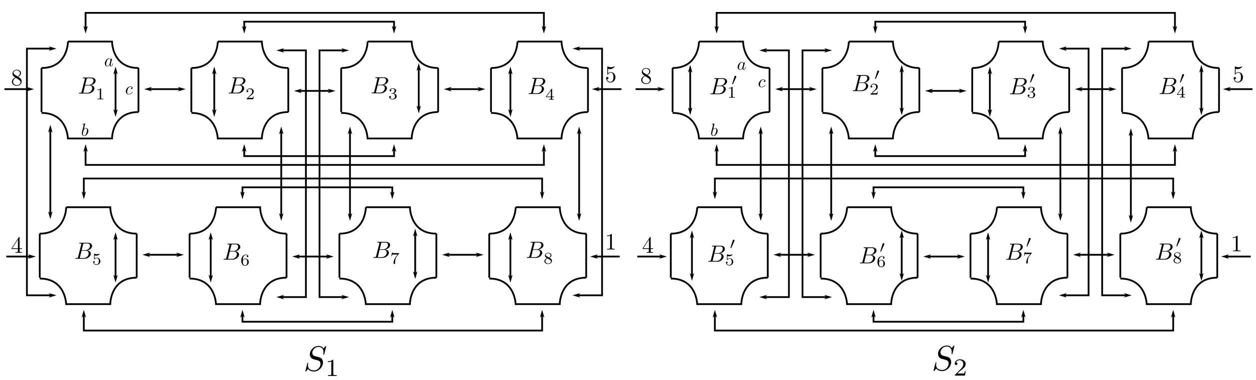

Figure 1: Gluing schemes for iso-length-spectral, non-isometric a genus 5 surfaces from [Bus86].

We now briefly summarize the idea behind showing Buser's genus 5 surfaces are iso-length-spectral but not isometric. We do not detail the higher genus cases, but the ideas are similar with variations in how the building blocks are constructed.

Buser's surfaces are not isometric. The building blocks B_i $(1 \leq i \leq 8)$ have three sets of identical edges with lengths a, b and c (see Figure 1). By adding an extra restriction on the lengths b and c, Buser is able to conclude the systoles of S_1 and S_2 are exactly geodesics of length c:

Lemma 2.1 (Lemma 3.3, Proposition 3.4 of [Bus86])**.** *Let $0 < c < b < 1$. Then any geodesic curve δ of B_i which connects two sides of B_i has length $\ell(\delta) \geq c$. Equality holds only if δ is a side (of length c) of a building block.*

As a result, S_1 and S_2 each have sets of four systoles $\{\gamma_i\}$ and $\{\gamma_i'\}$ $(1 \leq i \leq 4)$ respectively of length c. In S_1, the γ_i are between B_i and B_{i+1} for $i \in \{1, 3, 5, 7\}$. On the other hand, in S_2, the γ_i' are between B_i' and B_{i+1}' for $i \in \{2, 4, 6, 8\}$ (where, as always, $i + 1$ is taken mod 8). One can easily check that cutting along the multicurve $\{\gamma_i\}$ in S_1 yields a single connected component that is topologically a torus with 8 boundary components, while cutting along $\{\gamma'_i\} \subset S_2$ yields two connected components, each of which is topologically a torus with four boundary components. If S_1 and S_2 were isometric, cutting along their systoles would yield homeomorphic connected components, which is clearly not the case here.

Buser's surfaces are iso-length-spectral. Next, for every closed geodesic in S_1, Buser constructs one in S_2 with the same length (and vice versa) using a technique he calls *transplantation.* Given a closed geodesic $\gamma \subset S_1$ with a starting point p in the interior of some building block $B_i \subset S_1$, Buser specifies which building block $B'_{k(i)} \subset S_2$ one should start constructing $\gamma' \subset S_2$ in so that $\ell(\gamma) = \ell(\gamma')$. He uses the following set of rules, which depend on the parities of $\#a$ and $\#b$, defined below:

Algorithm 2.2 (5.3 (Initiation), [Bus86])**.** Let $\#a$ and $\#b$ be the number of times a curve γ transversely crosses sides of lengths a and b respectively, and let B_{n_1} (resp. $B'_{n'_1}$) denote the building blocks in S_1 (resp. S_2) in which to initiate γ (resp. γ').

1. If $\#a$ is even, take $n_1 = n'_1$;
2. If $\#a$ is odd and $\#b$ is even, take $n'_1 = n_1 + 1$;
3. If $\#a$ and $\#b$ are both odd, take $n'_1 = n_1 + 2$.

To construct γ', one simply decomposes γ into geodesic segments $\{\gamma_j\}_{j=1}^N$ such that each γ_j is contained completely in some building block B_j and γ_1 and γ_N are contained in the same building block and share an endpoint at p. Then, for each γ_j, taking advantage of the fact that the building blocks in S_1 and S_2 are all identical, one can construct a $\gamma'_j \subset B'_j$ identical to each $\gamma_j \subset B_j$. Buser shows that following Algorithm 2.2, the set of $\{\gamma'_j\}$ will always close up to a geodesic loop. Furthermore, since $\ell(\gamma_j) = \ell(\gamma'_j)$ for all j, it follows that $\ell(\gamma) = \ell(\gamma')$, as desired. The same set of initiation rules also applies in the other direction (constructing a geodesic in S_1 given one in S_2).

2.2 Buser's techniques in the surface amalgam setting

We now specify some notation and establish some facts used in the iso-length-spectrality proofs in the remainder of the paper. The results and definitions in this section are either heavily inspired by or taken directly from [Bus86].

Let β_i (resp. β'_i) be a connected geodesic segment in S_1 (resp. S_2) which can be written as a union $\bigcup_{k=1}^{L} \gamma_{i,k}$ (resp. $\bigcup_{k=1}^{L} \gamma'_{i,k}$) of segments each completely contained in a single building block. We will see later that every closed geodesic in a surface amalgam constructed in this paper can be written as a concatenation of β_i's or β'_i's. We define

$$\delta_i(k) = n'_{i,k} - n_{i,k} \pmod 8, \tag{1}$$

where $n_{i,k}$ (resp. $n'_{i,k}$) is the index of the building block containing $\gamma_{i,k}$ (resp. $\gamma'_{i,k}$). Thus, when we say "initiate β'_i with the rule $\delta_i(0) = N$," we mean that we will set $n'_{i,0}$ equal to $n_{i,0} + N$. In other words, if β_i starts in $B_{n_{i,0}} \subset S_1$, then β'_i will start in $B'_{n_{i,0}+N} \subset S_2$.

For the convenience of the reader, we list observations from Section 5 of [Bus86] used to prove the validity of Algorithm 2.2, which may be checked. We will also use these observations extensively in iso-length-spectrality proofs in the remainder of the paper. Note that while Buser works with closed surfaces, the segments β_i and β_i' are contained entirely within closed surfaces, so the observations are still applicable in the setting of surface amalgams.

Lemma 2.3 (c.f. Proof of 5.3 in [Bus86]). *Let δ_i, β_i, and β_i' be as above. Then the following changes to δ_i are observed whenever β_i and β_i' cross edges of building blocks of S_1 and S_2 respectively:*

1. Crossing a side of length a. *If $\delta_i(k)$ is even, then $\delta_i(k+1) = \delta_i(k) + 4 \pmod 8$. If $\delta_i(k)$ is odd, then $\delta_i(k+1) = \delta_i(k)$.*
2. Crossing a side of length b. *If $\delta_i(k) = 0$ or 4, $\delta_i(k+1) = \delta_i(k)$. If $\delta_i(k) = \pm 2$, $\delta_i(k+1) = \delta_i(k) + 4 \pmod 8$. There are other possible scenarios, but only these are used in this paper.*
3. Crossing a side of length c. *Regardless of the value of $\delta_i(k)$, $\delta_i(k+1) = \delta_i(k)$.*

2.2.1 Translated and transplanted copies

Next, we define *translated copies* of curves, which, roughly speaking, are locally isometric copies of a curve on the *same* surface (either S_1 or S_2). In contrast, a *transplanted copy* of a curve in S_1 is a curve on the *other* surface S_2 which has the same length (and vice versa). To formalize this, we first present a definition from [Bus86]:

Definition 2.4 (c.f. Definition 5.2, [Bus86]). Let β and β' be two curves in S_1 or S_2 which have the same length. Suppose $\beta = \bigcup_{k=1}^{L} \beta_k$, where each β_k is a curve contained entirely in a single building block. Similarly, suppose β' can be written as the union $\beta' = \bigcup_{k=1}^{L} \beta_k'$. Then β and β' are *locally congruent* if there exist local isometries $\{\varphi_k\}_{k=1}^{L}$ such that $\varphi_k(N(\beta_k)) = N(\beta_k')$ for every $k \in [1, L]$, where $N(\beta_k)$ and $N(\beta_k')$ are neighborhoods of β_k and β_k' respectively.

We can then define translated copies:

Definition 2.5 (Translated copies of geodesic segments). Given a geodesic segment $\beta = \bigcup_{k=1}^{L} \beta_k$ in S_1, a *translated copy* of β is a curve $\alpha = \bigcup_{k=1}^{L} \alpha_k \subset S_1$ which is locally congruent to β. We also require that if β begins (resp. ends) on an edge with a certain label, then α begins (resp. ends) on an edge with the same label. Furthermore, if β_k and α_k are contained in the building blocks $B_{n(k)}$ and $B_{m(k)}$ respectively, then for every $1 \le k \le L-1$, $B_{m(k)}$ meets $B_{m(k+1)}$ along an edge with the same label as that of the edge where $B_{n(k)}$ and $B_{n(k+1)}$ meet. One can also replace S_1 in the definition with S_2.

Following terminology from Section 11.6 of [Bus92], we also define the following.

Definition 2.6 (Transplanted copies of geodesic segments). Given a geodesic segment $\beta = \bigcup_{k=1}^{L} \beta_k$ in S_1, a *transplanted copy* of β is a curve $\beta' = \bigcup_{k=1}^{L} \beta_k' \subset S_2$ which is locally congruent to β. We also require that if β begins (resp. ends) on an edge with a certain label, then β' begins (resp. ends) on an edge with the same label. Furthermore, if β_k and β_k' are contained in the building blocks $B_{n(k)} \subset S_1$ and $B_{n(k)}' \subset S_2$ respectively, then for every $1 \le k \le L-1$, $B_{n(k)}'$ meets $B_{n(k+1)}'$ along an edge with the same label as that of the edge where $B_{n(k)}$ and $B_{n(k+1)}$ meet.

3 Proof of Theorem 1.1

We now proceed with our proofs of the existence of iso-length-spectral, non-isometric examples.

3.1 Homeomorphic, non-isometric surface amalgams

We first construct two homeomorphic, iso-length-spectral, non-isometric surface amalgams.

Construction 3.1. Consider S_1 and S_2 from [Bus86]. In each surface, consider the two closed geodesics of length $2a$ obtained from concatenating the top right edge of length a on B_2 (resp. B_6) with the top left edge of length a on B_3 (resp. B_7). We do the same for the bottom edges. We then identify all the red edges of length a on B_2 and B_6 (resp. B_2' and B_6') from Figure 2 and all blue edges on B_3 and B_7 (resp. B_3' and B_7').

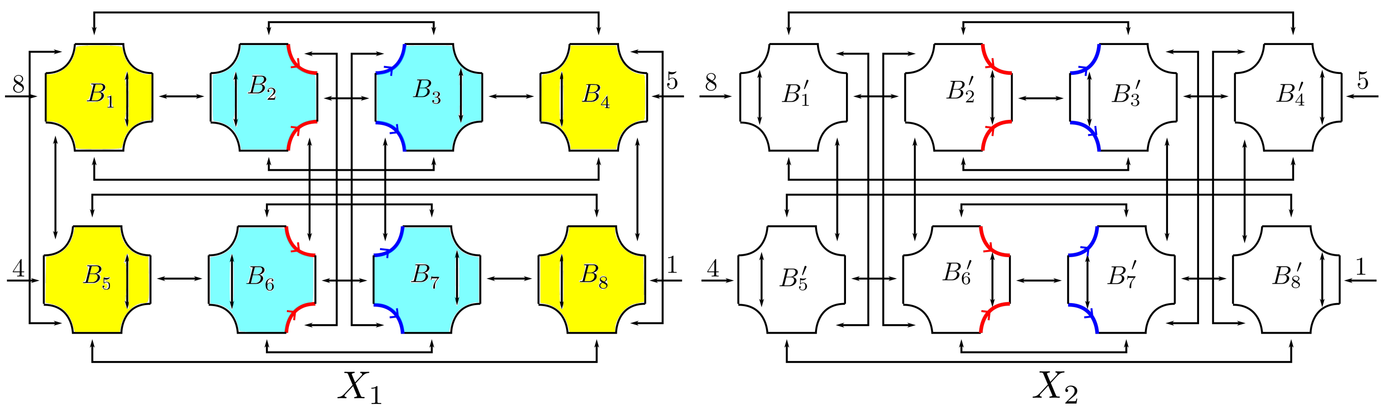

Figure 2: Homeomorphic, non-isometric, iso-length-spectral surface amalgams. The gluing curve consists of two closed geodesics of length $2a$ identified together. Orientations of the gluing curves are specified in the figure. Geodesic segments that are the same colors in X_1 are identified; the same is true for X_2. Cutting along the gluing geodesics and the systoles yields two connected components (in yellow and blue) for X_1 but only one connected component for X_2.

Proposition 3.2. *The surface amalgams X_1 and X_2 from Construction 3.1 are iso-length-spectral and homeomorphic, but not isometric.*

Proof. X_1 **and** X_2 **are homeomorphic.** Note that cutting along the four geodesics that are identified will yield a genus 3 surface with four boundary components of length $2a$ for both S_1 and S_2. The two loops of length $2a$ are nonseparating for both S_1 and S_2 so cutting along them yields homeomorphic surfaces $S_{3,4}$. Since there is bijection between homeomorphic chambers of X_1 and X_2, they are homeomorphic due to [Laf07].

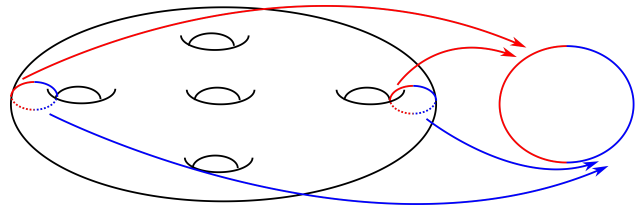

Figure 3: X_1 and X_2 are both homeomorphic to the above surface amalgam, created from identifying two non-separating closed geodesics on Buser's genus 5 example.

X_1 and X_2 are not isometric. In order for two surface amalgams to be isometric, there must be a bijection between isometric chambers due to the bijective correspondence between homeomorphic chambers (see [Laf07]). As with the examples in [Bus86], the only systoles in C_1 and C_2, chambers in X_1 and X_2 respectively, are the four closed geodesics of length c. Cutting along these systoles yields two connected components for C_1 and one connected component for C_2; thus, C_1 and C_2 cannot be isometric.

X_1 and X_2 are iso-length-spectral. It suffices to consider geodesics that do not intersect the gluing curve, as [Bus86] already shows a 1-1 correspondence between closed geodesics completely contained in S_1 and S_2. Consider γ, a closed geodesic in X_1. Decompose γ into $\bigcup_{i=1}^{N} \beta_i$ so that each β_i is a continuous geodesic segment in the original surface S_1 with endpoints on the gluing curve. We say a closed geodesic in $\gamma_1 \subset X_1$ is *identical* to γ if γ_1 and γ begin at the same point $x \in X_1$ on the gluing curve, and γ_1 can be decomposed into $\bigcup_{i=1}^{N} \beta_i^1$, a union of translated copies of β_i in S_1 with endpoints on the gluing curve. There is a natural isometry φ between gluing geodesics in X_1 and those in X_2. We say $\gamma' \subset X_2$ is *identical* to $\gamma \subset X_1$ if γ' begins and ends at $\varphi(x) \in X_2$ while γ begins and ends at $x \in X_1$, and γ_2 can be decomposed into $\bigcup_{i=1}^{N} \beta_i'$ so that each β_i' is a transplanted copy of β_i (in the sense of [Bus86]) that begins and ends on the gluing curve in X_2. We will argue that the numbers of closed geodesics identical to γ in X_1 and X_2 are the same, which shows the length spectra are the same.

We now construct closed geodesics identical to γ in X_1 by concatenating *admissible* translated copies of each β_i, which we define precisely below:

Definition 3.3. We say that a choice of geodesic segment β_i, where $2 \leq i \leq N$, is *admissible* if it is compatible with the previous choices of β_j $(j < i)$ in the following sense:

1. $\beta_{i-1} \cup \beta_i$ *does not backtrack*: β_i does not start on the same side of the building block that β_{i-1} ends on (see (a) on Figure 4), *and* if $i = N$, β_i does not end on the same side of the building block β_1 starts on;

2. $\beta_{i-1} \cup \beta_i$ *is continuous*: β_i begins at the same point on the gluing curve that β_{i-1} ends on (see (b) on Figure 4), *and*, if $i = N$, β_i ends at the same point on the gluing curve that β_1 begins at.

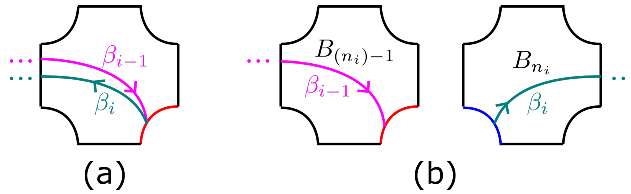

Figure 4: Two examples of (parts of) inadmissible β_i's. In (a), $\beta_{i-1} \cup \beta_i$ is a backtracking geodesic segment. In (b), β_{i-1} and β_i begin at different points of the gluing curve, as the red and blue segments are not identified, which means $\beta_{i-1} \cup \beta_i$ is disconnected.

We first prove the following:

Claim. *In X_1, there is exactly one translate of $\beta_i^1 := \beta_i$, β_i^2, which begins and ends at the same point on the gluing curve β_i begins and ends at.*

Proof of claim: If β_i begins on the top right (resp. top left, bottom right, or bottom left) side of length a in B_i, initiate β_i^2 at the same point on the top right (resp. top left, bottom right, or bottom left) side of length a in B_{i+4}. Let $\beta_i^j = \bigcup_{k=0}^{L} \gamma_{i,k}^j$ be a decomposition of β_i^j into geodesic segments that are contained entirely in a single building block, where $\gamma_{i,k}^j \subset B_{n_{i,k}^j}$ and $j = 1, 2$. Let $\epsilon_k = n_{i,k}^2 - n_{i,k}^1$. Note in particular that $\epsilon_0 = 4$. We observe the following changes to ϵ_j in the following scenarios:

- *β_i^1 and β_i^2 cross sides of length c.* In this case, $n_{i,k+1}^j = n_{i,k}^j \pm 1$ for both $j = 1, 2$ depending on whether the left or right side of a building block is crossed. Note that when crossing edges of length c, β_i^1 and β_i^2 either both cross left sides or both cross right sides. Then $\epsilon_{k+1} = (n_{i,k}^2 \pm 1) - (n_{i,k}^1 \pm 1) = n_{i,k}^2 - n_{i,k}^1 = \epsilon_k$.

- *β_i^1 and β_i^2 cross sides of length a.* In this case, for both $j = 1, 2$, $n_{i,k+1}^j = n_{i,k}^j$ or $n_{i,k}^j + 4$, depending on the parity of the building block. Provided that $\epsilon_k = 4 \pmod 8$, either $\epsilon_{k+1} = n_{i,k}^2 - n_{i,k}^1 = \epsilon_k$ or $\epsilon_{k+1} = (n_{i,k}^2 + 4) - (n_{i,k}^1 + 4) = \epsilon_k$.

- *β_i^1 and β_i^2 cross sides of length b.* In this case, for both $j = 1, 2$, $n_{i,k+1}^j = n_{i,k}^j \pm 1$ or $n_{i,k}^j \pm 3$, depending on the label of the building block. But again, if $\epsilon_k = 4$, then $\epsilon_{k+1} = (n_{i,k}^2 \pm 1) - (n_{i,k}^1 \pm 1) = n_{i,k}^2 - n_{i,k}^1 = \epsilon_k$ or $\epsilon_{k+1} = (n_{i,k}^2 \pm 3) - (n_{i,k}^1 \pm 3) = n_{i,k}^2 - n_{i,k}^1 = \epsilon_k$.

From this, we conclude that in fact, $\epsilon_L = 4$. That is, if β_i^1 ends on an edge of the building block B_N, then β_i^2 ends on the same point in the corresponding edge of B_{N+4}. Since there are only two edges that are top right, top left, bottom right, or bottom left edges of building blocks which are identified to create a subarc of the gluing geodesic, there cannot be another translate of β_i^1 sharing beginning and end points with β_i^1. Thus, there is exactly one such β_i^2, as claimed. □

We now compute $C_1(\gamma)$, the number of closed geodesics in X_1 identical to γ. Fix a copy of β_1, say $\beta_1^1 \subset X_1$. By construction, β_1^1 can be concatenated with any of the two copies of β_2 in $\{\beta_2^j\}_{j=1}^2$ to create a geodesic segment *unless* some β_2^j is chosen so that $\beta_2^j \cup \beta_1^1$ backtracks, in which case there is only one admissible copy of β_2 which can be concatenated with β_1^1. Using the same logic for all $2 \leq i \leq N$, we have the following general fact:

$c_i := \#\{\beta_i^j$ which can be concatenated with some fixed copy of $\beta_{i-1}\}$

$$= \begin{cases} 1 \text{ if } \beta_{i-1} \text{ ends on a top red edge from Figure 2 and } \beta_i \text{ begins on a top red edge,} \\ \text{ or the statement is true if "red" is replaced with "blue" and/or "top" is replaced with "bottom";} \\ 2 \text{ otherwise.} \end{cases}$$

We now examine how to choose a copy of β_N from $\{\beta_N^j\}_{j=1}^2$. The set of admissible β_N now depends on our choice of β_{N-1}^j and β_1^1, and specifically on whether the following facts are true:

- *Fact 1:* β_{N-1}^j ends on a top red edge and β_N begins on a top red edge (or the statement is true if we replace "red" with "blue" and/or "top" with "bottom");
- *Fact 2:* β_N ends on a top red edge and β_1 begins on a top red edge (or the statement is true if we replace "red" with "blue" and/or "top" with "bottom").

We now do some casework, depending on whether Facts 1 and 2 are satisfied.

Case One: Neither Fact 1 nor Fact 2 is true. In this case, any choice of pairs of translates of β_1 and β_N is compatible. Thus, each choice of β_1^j $(j = 1, 2)$ has $\prod_{i=2}^{N} c_i = \left(\prod_{i=2}^{N-1} c_i\right)(2)$ choices of sequences of admissible translates of β_i. Then $C_1(\gamma) = 2\left(\prod_{i=2}^{N-1} c_i\right)(2)$. See Figure 5.

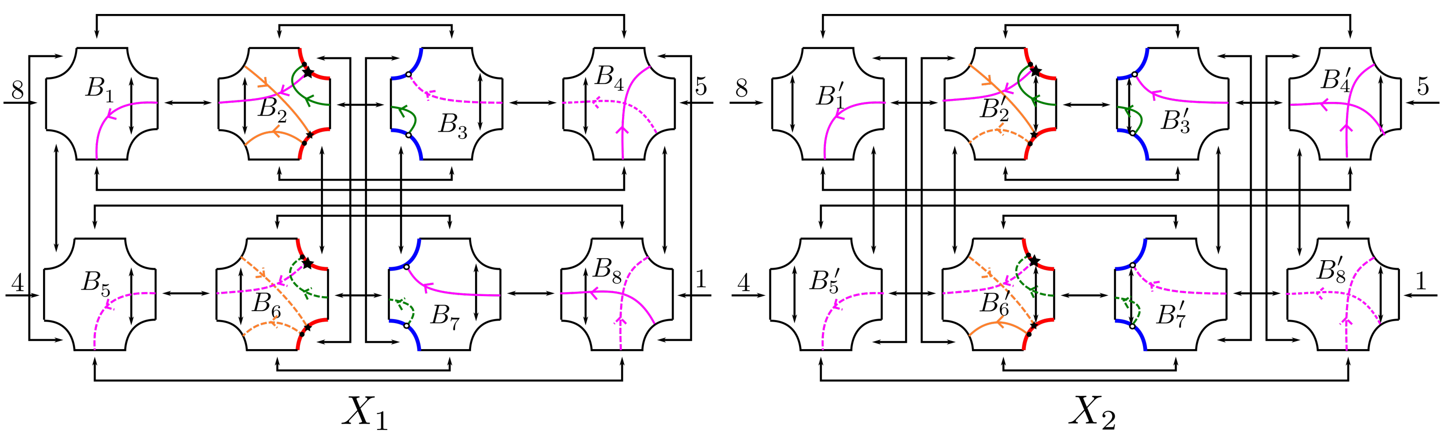

Figure 5: An illustration of Case 1, with identical copies of closed geodesics created by concatenating the pink (β_1), green (β_2), and orange (β_3) geodesic segments. Two closed geodesics are illustrated in each of X_1 and X_2, one solid and one dashed. The star icon indicates the start of each geodesic. Here, $N = 3$ and $c_2 = 2$, so $C_1(\gamma) = 2(2)(2) = C_2(\gamma)$. We can check that there are indeed eight identical, non-backtracking closed geodesics in each of X_1 and X_2.

Case Two: One of Fact 1 or Fact 2 is true. Suppose Fact 1 is true. Then any choice of β_N^j is compatible with a fixed choice of translate of β_1, but there is only one admissible translate of β_N for each fixed translate of β_{N-1}. Suppose Fact 2 is true. Again, given a fixed β_1^j, there is only one admissible translate of β_N, as one of them is not compatible with β_1^j. In either case, we slightly modify the equation for $C_1(\gamma)$ from the previous case: $C_1(\gamma) = 2\left(\prod_{i=2}^{N-1} c_i\right)(1)$. See Figure 6.

Case Three: Both Fact 1 and Fact 2 are true. This case is the most nuanced, but surprisingly, the count is the same as in Case 2. There are four possible pairs of geodesic segments β_1^j and

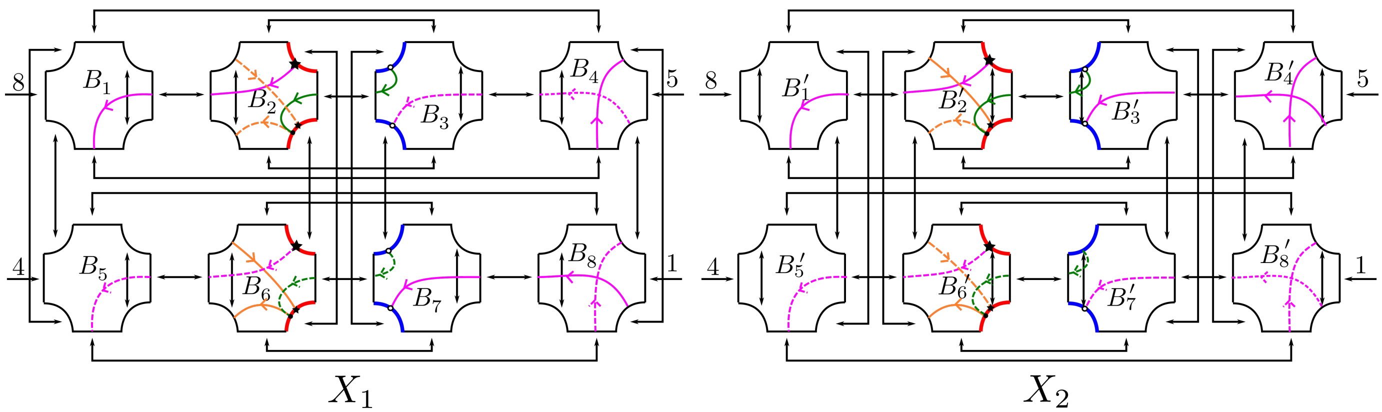

Figure 6: An illustration for Case 2. Again, two closed geodesics are shown, one solid and one dashed, and the star indicates the starting point of each geodesic. Here, $N = 3$ and $c_2 = 2$, so $C_1(\gamma) = 2(2) = C_2(\gamma)$. We can check that there are four identical, non-backtracking closed geodesics in each of X_1 and X_2.

β^k_{N-1} $(j, k \in \{1, 2\})$ which satisfy (2) of Definition 3.3. For exactly two of these pairs, there is one inadmissible copy of β_N: the translate of β_N with starting point coinciding with the endpoint of β^k_{N-1} and endpoint coinciding with the starting point of β^j_1. For all other pairs, there are two inadmissible (and thus no admissible) copies of β_N: one translate with endpoint coinciding with the starting point of β^j_1 and one (other) translate with starting point coinciding with the endpoint of β^k_{N-1}. Thus, $C_1(\gamma) = 2\left(\prod_{i=2}^{N-1} c_i\right)(1) + 2\left(\prod_{i=2}^{N-1} c_i\right)(0) = 2\left(\prod_{i=2}^{N-1} c_i\right)$. See Figure 7.

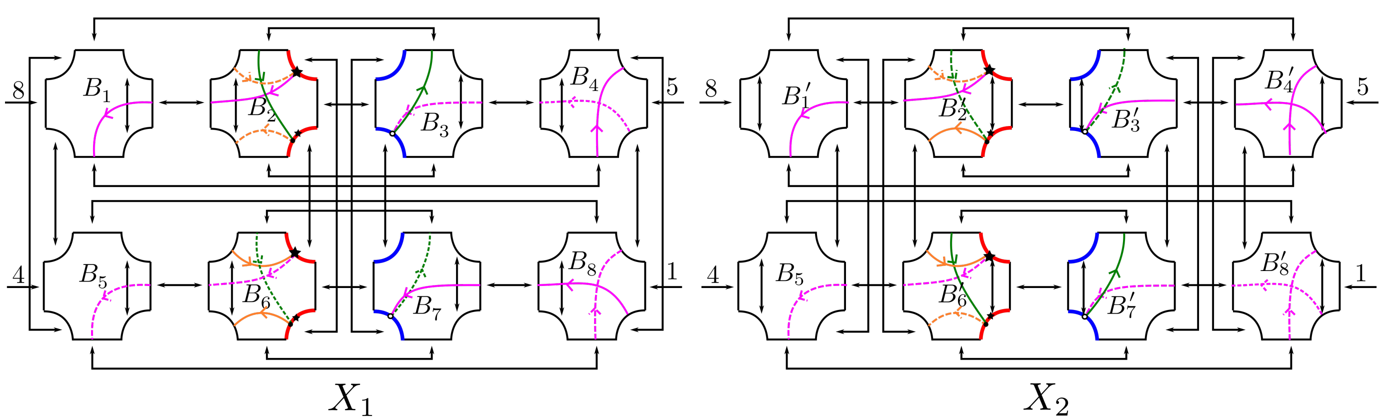

Figure 7: An illustration for Case 3. Again, $N = 3$ and $c_2 = 2$, so $C_1(\gamma) = 2(2) = C_2(\gamma)$. We can check that there are indeed four identical, non-backtracking closed geodesics in each of X_1 and X_2.

We now construct an identical transplanted copy of each β^j_i, $\beta'^j_i = \bigcup_{k=0}^{L} \gamma'^j_{i,k}$, where $\gamma'^j_{i,k} \subset B'_{n'^j_{i,k}}$ and β'^j_i is initiated with the rule $\delta^j_i(0) := n'^j_{i,0} - n^j_{i,0} = 0$ (see Equation (1)). We claim that β^j_i meets the gluing curve if and only if β'^j_i intersects a gluing curve. By Lemma 2.3, if $\delta^j_i(k) = 0$ or 4, crossing a side of length c or b does not change δ^j_i and crossing a side of length a changes δ^j_i to $\delta^j_i + 4$. Thus, $\delta^j_i(k) = 0$ or 4 for all $0 \le k \le L$; that is, $n^j_{i,k}$ and $n'^j_{i,k}$ differ by either 0 or 4 (see, for example, any of the previous 3 figures). Then β^j_i meets a gluing curve if and only if β'^j_i

does. This establishes a natural bijection between $\{\beta_i^j\}_{j=1}^2$ and $\{\beta'^j_i\}_{j=1}^2$, so one can make the same computations as before for calculating $C_2(\gamma)$, which counts the number of identical copies of γ in X_2. We obtain in the end that $C_1(\gamma) = C_2(\gamma)$. One can therefore bijectively map copies of γ in X_1 to those in X_2. □

3.2 Non-homeomorphic, iso-length-spectral surface amalgams

We first remind the reader of a convenient way to determine whether two simple, thick surface amalgams are homeomorphic, using a criterion established by Lafont in [Laf07].

Proposition 3.4 (Corollary 3.4 of [Laf07])**.** *If $f : \widetilde{X_1} \to \widetilde{X_2}$ is a quasi-isometry, then f induces a bijection between homeomorphic chambers of X_1 and X_2.*

Recall that if two compact metric spaces are homeomorphic, then there is a quasi-isometry between their universal covers by the Milnor-Schwarz Lemma. From Proposition 3.4, it then follows that the chambers of two homeomorphic simple, thick surface amalgams are necessarily in bijective correspondence with each other. With this in mind, we now construct two isopectral, non-homeomorphic hyperbolic surface amalgams.

Construction 3.5. Consider S_1 and S_2 from [Bus86] which have sets of systoles $\{\gamma_i\}_{i=1}^4$ and $\{\gamma_i'\}_{i=1}^4$. Construct X_1 by identifying all the systoles $\gamma_i \subset X_1$ according to the orientations specified in Figure 9. Similarly, glue together all the $\gamma_i' \subset S_2$ to construct X_2.

Proposition 3.6. *The surface amalgams X_1 and X_2 from Construction 3.5 are iso-length-spectral but not homeomorphic.*

Proof. **X_1 and X_2 are not homeomorphic.** Cutting along the gluing curves yields one chamber for X_2 and two for X_1, so by Proposition 3.4 it is impossible to establish a bijection between the two collections of chambers (see Figure 8 and Figure 9).

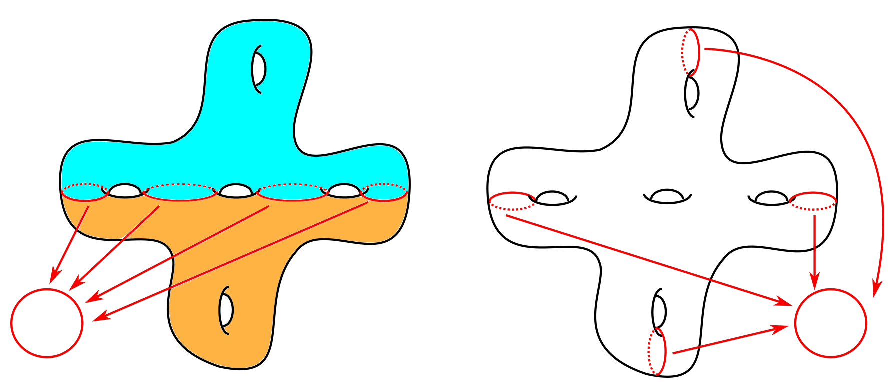

Figure 8: X_1 and X_2 are homeomorphic to the surface amalgams on the left and right respectively.

X_1 and X_2 are iso-length-spectral. We follow a similar strategy as before, by showing that the number of closed geodesics identical to a given closed geodesic $\gamma \subset X_1$ is the same in X_1 and X_2. Again, we focus our attention on closed geodesics that do not intersect the gluing curve. We decompose γ into a union $\bigcup_{i=1}^{N} \beta_i$ of geodesic segments that project to connected geodesic segments in S_1 beginning and ending on the gluing curve and count the number of ways to concatenate admissible translates of each β_i.

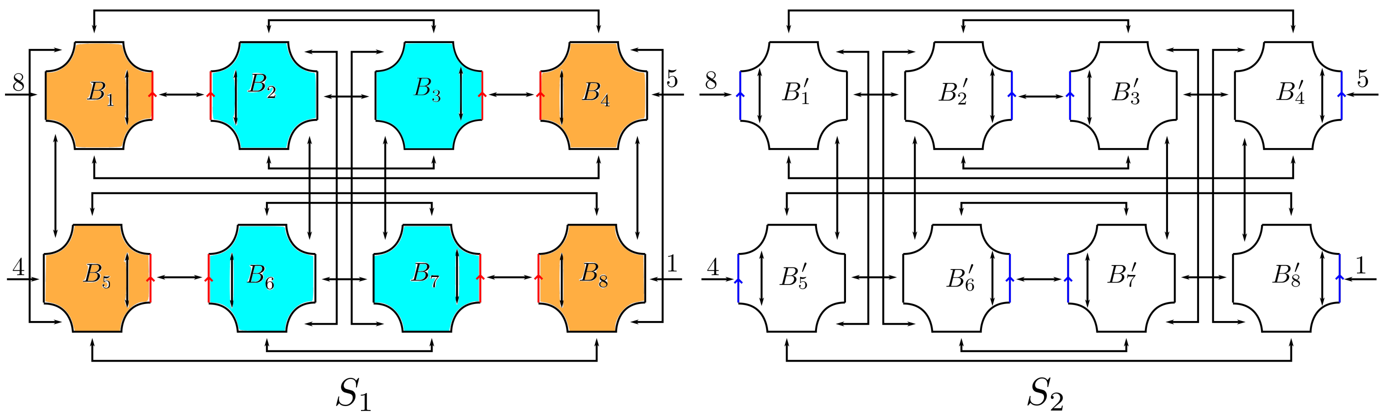

Figure 9: The two chambers of X_1 are in orange and blue while X_2 only has one chamber.

We will show that in X_1, there are four translates of β_i with the same beginning and end points on the gluing curve. If β_i begins on the right (resp. left) edge of a building block, we show there is one copy starting on the right edge (resp. left) of each odd (resp. even) labeled building block. In particular, we claim that each translated copy of β_i will end on a building block with the same parity as the one β_i ends on (and thus share an endpoint with β_i).

Let $\{\beta_i^j\}_{j=1}^4$ be the set of geodesic segments in X_1 which are translated copies of β_i that either all begin on the left sides of the even labeled building blocks or all begin on the right sides of the odd labeled building blocks, depending on where β_i begins. Suppose $\beta_i^1 := \beta_i$. We check that β_i^j, where $j = 2, 3$, or 4, ends on a building block whose label has the same parity as the one β_i^1 ends on, allowing it to be concatenated with the next geodesic segment. Let $\beta_i^j = \bigcup_{k=0}^{L} \gamma_{i,k}^j$, where each γ_k^j is contained in a single building block, $B_{n_k^j}$. Define $d_i^j(k) := n_{i,k}^j - n_{i,k}^1$. Note that $d_i^j(0)$ is even since each β_i^j must start on either an odd or even labeled building block. Crossing a side of length c will not change $d_i^j(k)$. Furthermore, crossing a side of length b (resp. a) will change (resp. not change) the parity of both $n_{i,k}^j$ and $n_{i,k}^1$, so the parity of $d_i^j(k)$ remains the same for all $0 \leq k \leq L$. In particular, $d_i^j(L)$ is even, so the parity of $n_{i,L}^j$ will match that of $n_{i,1}^j$.

We can now compute $C_1(\gamma)$ using the same strategy as before. Using the same argument from Section 3.1 (replacing 1 and 2 with 3 and 4 respectively), we deduce that for each $2 \leq i \leq N-1$:

$$c_i := \#\{\beta_i^j \text{ which can be concatenated with some fixed copy of } \beta_{i-1}\}$$
$$= \begin{cases} 3 \text{ if } \beta_{i-1} \text{ ends on a right (resp. left) edge, and } \beta_i \text{ begins on a right (resp. left) edge;} \\ 4 \text{ otherwise.} \end{cases}$$

We then calculate c_N depending on whether the following facts are true:

- *Fact 1:* β_{N-1} ends on a right edge and β_N begins on a right edge (or the statement is true if we replace "right" with "left");
- *Fact 2:* β_N ends on a right edge and β_1 begins on a right edge (or the statement is true if we replace "right" with "left").

By the same proof as before but replacing $0, 1$ and 2 with $2, 3,$ and 4 respectively, we have $C_1(\gamma) = 4\left(\prod_{i=2}^{N-1} c_i\right)(4)$ for Case 1, $C_1(\gamma) = 4\left(\prod_{i=2}^{N-1} c_i\right)(3)$ for Case 2, and $C_1(\gamma) = \frac{4}{16}(4)\left(\prod_{i=2}^{N-1} c_i\right)(3)+$

$\frac{12}{16}(4)\left(\prod_{i=2}^{N-1} c_i\right)(2) = \left(\prod_{i=2}^{N-1} c_i\right)(3)+3\left(\prod_{i=2}^{N-1} c_i\right)(2)$ for Case 3. The fractions from Case 3 come from the fact that there are now a total of 16 choices of pairs of geodesic segments β_1^j and β_{N-1}^k where $j, k \in \{1, 2, 3, 4\}$. For exactly four of these pairs, there is exactly one inadmissible copy of β_N which begins on the same building block as the end of β_{N-1}^k and ends on the same building block as the beginning of β_1^j. For the 12 other pairs, there are two inadmissible copies of β_N.

We now show $C_2(\gamma) = C_1(\gamma)$. First, we claim there are four transplanted copies of β_i that begin on the right (resp. left) side of each odd (resp. even) labeled building block and end at the same point. There is a natural isometry between gluing geodesics in X_1 and those in X_2; if a gluing geodesic c_n is the right edge of B_n and the left edge of B_{n+1}, let $\varphi(c_n)$ be the right edge of B_{n+1} and the left edge of B_{n+2}. If each β_i^j begins at $x \in X_1$, we construct a set $\{\beta'^j_i\}_{j=1}^4 = \left\{\bigcup_{k=0}^{L} \gamma'^j_{i,k}\right\}_{j=1}^4$ of transplanted copies of β_i^j which begin at $\varphi(x) \in X_2$ and pass through building blocks $B'_{n'^j_{i,k}}$. Initiate each $\beta'^j_{i,k}$ with the rule $\delta_i^j(0) := n'^j_{i,0} - n^j_{i,0} = 1$. By Lemma 2.3, if $\delta_i^j(k) = \pm 1$, crossing sides of length a or c does not affect $\delta_i^j(k)$ while crossing a side of length b sends ± 1 to ∓ 1. Regardless of the sequences of edges crossed, $\delta'^j_i(L) = 1$ or -1 for all j. Thus, each β'^j_i will end on the same side of a gluing geodesic that each β_i^j ends on. Each β'^j_i will also end on a distinct building block.

Thus, as before, one can make the same computations for $C_2(\gamma)$ that counts the number of identical copies of γ in X_2. We obtain in the end that $C_1(\gamma) = C_2(\gamma)$. As a result, one can bijectively map copies of γ in X_1 to copies of γ in X_2. □

Remark 3.7. We remark that X_1 and X_2 are commensurable. Indeed, they have a common double cover. The chambers in X_1, which are both tori with four boundary components, lift to their double covers, which are tori with 8 boundary components, $S_{1,8}$. In other words, the chambers in X_1 each lift to a copy of the chamber in X_2. In summary, X_1 and X_2 both have double covers consisting of two copies of $S_{1,8}$ and two gluing curves obtained from identifying two quartets of lifts of boundary components originally identified in X_1 and X_2.

4 Proof of Theorem 1.3

We now prove Theorem 1.3, our second main result following the construction below.

Construction 4.1. Consider, again, the surfaces from [Bus86]. We will glue three copies of each surface together to create X_1 and X_2. For both X_1 and X_2, there will be one gluing curve which will consist of unions of perpendiculars between edges of length b that bisect all the building blocks except ones labeled 6 and 7.

Note that in each surface, three pairs of perpendiculars form geodesics: the ones in building blocks with labels 1 and 4, 2 and 3, and 5 and 8. As a result, we need to specify how the geodesics are glued together. We will identify the halves of geodesics on the odd numbered building blocks and halves of geodesics on even numbered building blocks. See Figure 10 for the orientations of the gluing curves.

X_1 **and** X_2 **are weak length isospectral.** We use Buser's transplantation technique. As before, it suffices to consider closed geodesics that are not closed geodesics in copies of S_1 and S_2. We decompose $\gamma \subset X_1$ into a union of geodesic segments $\bigcup_{j=1}^{N} \gamma_j$ each of which is contained entirely within a building block B_{n_j}. We must specify an algorithm to construct a transplanted copy of γ,

$\gamma' \subset X_2$, which is a union $\bigcup_{k=1}^{N} \gamma'_k$ of geodesic segments that are each completely contained within a building block in S_2, start and end on the gluing curve in X_2, and do not backtrack. We will label the copies of S_1 in X_1 with S_1^1, S_1^2, and S_1^3 and similarly label the copies of S_2 in X_2. In the following algorithm, we will write $\delta^s(j) = {n'}_j^s - n_j^s$, which gives instructions for transplanting geodesic segments from S_1^s to S_2^s.

Algorithm 4.2. Suppose $\gamma_{j-1} \cup \gamma_j$ does not project to a connected geodesic segment in any copy of S_1. Moreover, suppose that following Algorithm 2.2, $\beta_{j,j+L} := \bigcup_{i=0}^{L} \gamma_{j+i}$ projects to a continuous geodesic segment in some copy of S_1, S_1^s, where $1 \le s \le 3$. We then initiate γ'_j using the following rules (unless otherwise specified, γ'_j will be initiated on S_2^s):

1. If γ_j lies in a building block indexed by an element in the set $\{1, 4, 5, 8\}$, we set $\delta^s(j) = 0$ if $\#b$ is even and 4 if $\#b$ is odd.

2. If γ_{j+L} lies in a building block indexed by an element in the set $\{1, 4, 5, 8\}$, we set $\delta^s(j) = 0$.

3. Otherwise, set $\delta^s(j) = 2$.

4. If any of the previous rules cause backtracking, copy γ'_j over to another copy of B'_{n_j} (e.g. ${B'}_{n_j}^t \subset S_2^t$ where $s \neq t$).

Remark 4.3. Three surfaces are needed in order to ensure (4) from Algorithm 4.2 can always be applied in order to prevent backtracking. In fact, only two surfaces are needed to correct for backtracking when constructing γ'_j, where $j < N$, but for $j = N$, one needs to ensure that γ'_N is admissible with respect to both γ'_1 and γ'_{N-1}.

We now show that following (1), (2) and (3) from Algorithm 4.2, for each $\beta_{j,j+L} \subset S_1^s$, we can obtain a geodesic segment $\beta'_{j,j+L} \in S_2^s$ which begins and ends on the gluing curve. If $\gamma_j \subset B_{n_j}$ for some $n_j \in \{1, 4, 5, 8\}$, then we set $\delta^s(j) = 0$ if $\#b$ is even and $\delta^s(j) = 4$ if $\#b$ is odd so that regardless, $\gamma'_j \subset B'_{n'_j}$, where $n'_j = k + 4 \in \{1, 4, 5, 8\}$. By Lemma 2.3, if $\delta^s = 0$ or 4, crossing a side of length a or c leaves δ^s invariant, while crossing a side of length b replaces δ^s by $\delta + 4$. Thus, $\delta^s(j + L) = 0$, so $B^s_{n'(j+L)}$ intersects a gluing curve since $B^s_{n(j+L)}$ does. Suppose $\gamma_{j+L} \subset B_k$ where $k \in \{1, 4, 5, 8\}$. Then setting $\delta^s(j) = 0$ ensures that γ'_j begins on a building block intersecting a gluing curve and that $\delta^s(j+L)$ is 0 or 4. Thus, γ'_{j+L} lies in a building block indexed by an element of the set $\{1, 4, 5, 8\}$, which necessarily intersects the gluing curve.

This leaves the case ($\gamma_j \subset B_{n_j}$, where $n_j \in \{2, 3\}$) *and* ($\gamma_{j+L} \subset B_{n_{j+L}}$, where $n_{j+L} \in \{2, 3\}$). We set $\delta^s(j) = 2$. Since $n_j \in \{2, 3\}$, $n'_j \in \{1, 4, 5, 8\}$ and the building blocks indexed by this set all intersect the gluing curve. By Lemma 2.3, $\delta = \pm 2$, crossing a side of length a or b replaces δ by $\delta + 4$, while as before, crossing a side of length c does not change δ. Thus, if $\#a + \#b$ is even (resp. odd), then $\delta^s(j + L) = 2$ (resp. -2). Either way, since $n^s_{j+L} \in \{2, 3\}$, we have that ${n'}^s_{j+L} \in \{1, 4, 5, 8\}$, and the building blocks indexed by this set all intersect the gluing curve.

We need to check an additional condition to ensure $\gamma'_j \cup \gamma'_{j-1}$ is connected in X_2. It suffices to show if γ'_{j-1} ends in an odd (resp. even) numbered building block, γ'_j also begins in an odd (resp. even) numbered building block. We know the previous sentence to be true if γ' is replaced with γ. In all cases, $\delta^s(k)$ is even for any $j \le k \le j + L$. As a consequence, the parities of the building blocks containing γ^s_j and ${\gamma'}^s_j$ are the same. This is enough to prove what we want.

Remark 4.4. Unfortunately, Algorithm 4.2 does not yield a 1-1 correspondence between identical geodesic segments in X_1 and X_2, even if we restrict to single subsets of X_1 and X_2 consisting of a

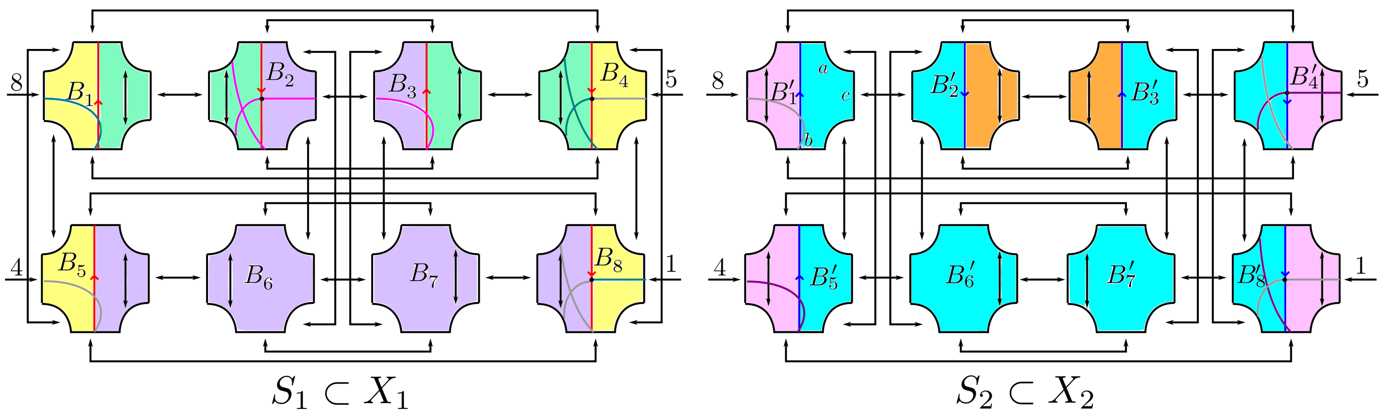

Figure 10: Weak length isospectral but noncommensurable X_1 and X_2 with the chambers shown in different colors. The gluing curves in X_1 are shown in red, while the ones in X_2 are dark blue. The pink and green geodesics in X_1 both map to the purple geodesic in X_2, so Algorithm 4.2 does not give a 1-1 correspondence between transplanted closed geodesics.

single copy of S_1 or S_2 with geodesics identified. Figure 10 illustrates this. Following Algorithm 4.2, both the pink and green curve in X_1 are assigned to the purple curve in X_2. Thus, one cannot show iso-length-spectrality using transplantation via Algorithm 4.2. In the picture, we also show all the identical closed geodesics that begin at the same point (indicated by black dots) in X_1 and X_2 respectively. Note that there are three closed geodesics in X_1 and only two in X_2, as initiating a geodesic from B'_2 does not result in a closed geodesic in X_2.

One may ask whether we can come up with initiation rules so that there is indeed a 1-1 correspondence between identical closed geodesics. Unfortunately, the answer is no. We show this by counting identical copies of a particular closed geodesic in X_1 and X_2.

We revisit the closed geodesics in Figure 10. In X_1, notice that on each surface S_1^s, there are three identical closed geodesics (depicted in green, pink, and gray) that begin and end at the same point $x \in X_1$, which is depicted as a black dot. In contrast, on each surface S_2^s in X_2, there are only two closed geodesics beginning and ending at the corresponding point $x \in X_2$. In $S_1^2 \subset X_1$, there are also two geodesics beginning and ending in the odd indexed building blocks, on the half of the gluing x is not on. In $S_2^s \subset X_2$, there are also two such closed geodesics; see Figure 11. Thus, in total, since there are 3 copies of each surface, there are $5(3) = 15$ copies of identical closed geodesics in X_1 but only $4(3) = 12$ copies of the same closed geodesics in X_2. This shows it is not possible to achieve a 1-1 correspondence between identical closed geodesics in X_1 and X_2; thus, one cannot prove iso-length-spectrality via transplantation. We remark that this does not rule out iso-length-spectrality of X_1 and X_2, which is improbable but still possible.

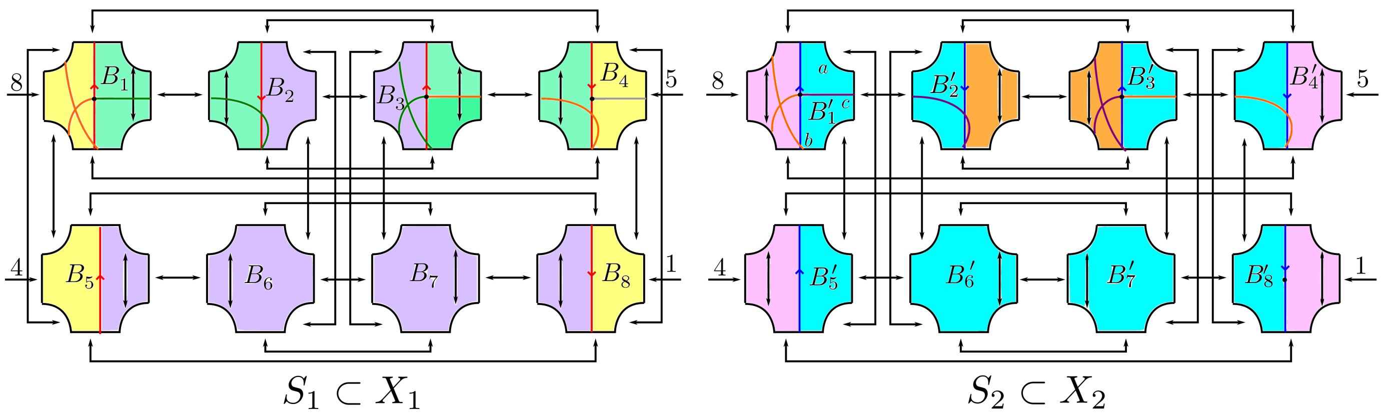

Figure 11: Copies of closed geodesics identical to the ones shown in Figure 10 which begin and end in the odd-labeled building blocks.

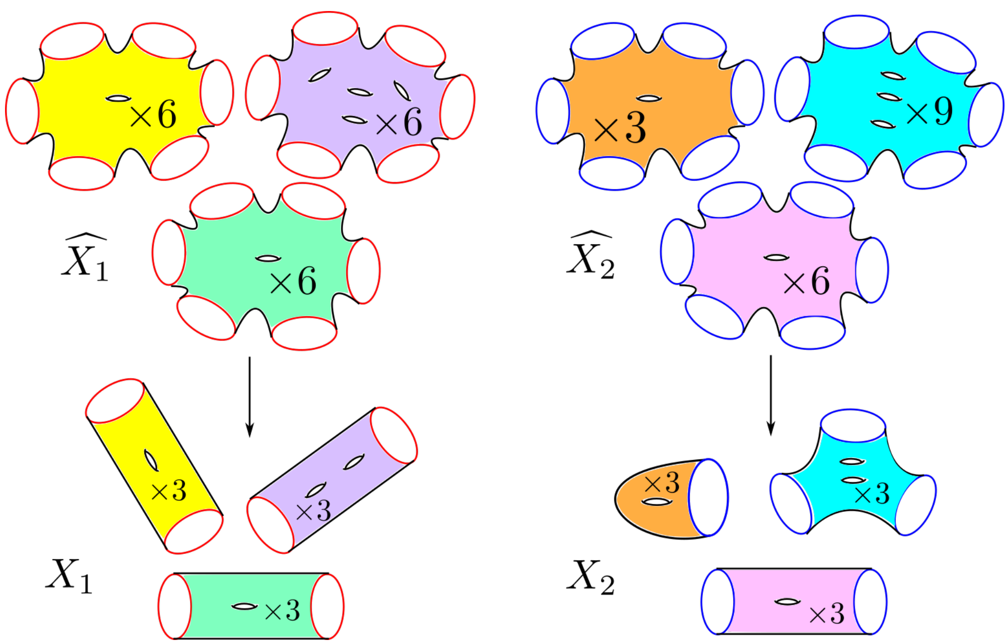

Figure 12: X_1 and X_2, which are constructed by gluing together the red and blue sets of boundary components, are not commensurable. Their six-sheeted covers referenced in the proof of Theorem 1.3 are also shown.

X_1 and X_2 are not commensurable. We now show that X_1 and X_2 do not have a common finite-sheeted cover. Note that X_1 has six tori with two boundary components, which we denote by $S_{1,2}$, and three genus two surfaces with two boundary components $S_{2,2}$; again, all the boundary components are identified. On the other hand, X_2 consists of three tori with one boundary component $S_{1,1}$, three tori with two boundary components $S_{1,2}$, and three genus two surfaces with three boundary components $S_{2,3}$, and all the boundary components are identified together.

Consider a six-sheeted cover of X_1, $\widehat{X_1}$, which consists of six copies of a genus four surface with six boundary components (three-sheeted covers of $S_{2,2}$) and 12 copies of a torus with six boundary components (three-sheeted covers of $S_{1,2}$). Consider also six-sheeted cover of X_2, $\widehat{X_2}$, which consists of nine copies of genus three surfaces with six boundary components (double covers of $S_{2,3}$) and nine copies of a torus with six boundary components (six three-sheeted covers of $S_{1,2}$ and three six-sheeted covers of $S_{1,1}$) (see Figure 12). For both $\widehat{X_1}$ and $\widehat{X_2}$, there are six gluing curves, and exactly one boundary component in each chamber is glued to each gluing curve. Note that the Euler characteristics of the chambers in $\widehat{X_1}$, in ascending order, are $\{\underbrace{-12, ..., -12}_{\times 6}, \underbrace{-6, ..., -6}_{\times 12}\}$ while those in $\widehat{X_2}$ are $\{\underbrace{-10, ..., -10}_{\times 9}, \underbrace{-6, ..., -6}_{\times 9}\}$. Let $\{S_i\}$ and $\{T_i\}$ denote the collections of chambers of $\widehat{X_1}$ and $\widehat{X_2}$ respectively, labeled so that $\chi(S_1) \leq \chi(S_2) \leq ... \leq \chi(S_{18})$ and $\chi(T_1) \leq \chi(T_2) \leq ... \leq \chi(T_{18})$. A generalization of Proposition 3.3.2 of [Sta17], proved in Section 5.2 of [DST18], implies that $\pi_1(\widehat{X_1})$ and $\pi_1(\widehat{X_2})$ are abstractly commensurable (e.g. they do not have isomorphic finite-index subgroups) if and only if all the $\dfrac{\chi(S_i)}{\chi(T_i)}$ are equal. Note, however, that $\dfrac{\chi(S_1)}{\chi(T_1)} = \dfrac{-12}{-10} \neq \dfrac{-6}{-6} = \dfrac{\chi(S_{18})}{\chi(T_{18})}$. It then follows that $\pi_1(\widehat{X_1})$ and $\pi_1(\widehat{X_2})$ are not abstractly commensurable.

Since abstract commensurability is an equivalence relation, $\pi_1(X_1)$ and $\pi_1(X_2)$ also cannot be abstractly commensurable since $\pi_1(X_i)$ is abstractly commensurable to $\pi_1(\widehat{X_i})$ for $i = 1, 2$. By the Galois correspondence of covering spaces for CW complexes, it then follows that X_1 and X_2 also cannot be commensurable, as claimed. □

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