REU 2017

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July 2017

1 Introduction

This is a write-up for the Georgia Tech Summer 2017 REU group that studied mapping class groups and train tracks.

We can think of a symmetry of a surface as a "mixing" of the surface. For example, a taffy puller mixes taffy through the symmetric pulling action of its arms. We are interested in how we can measure the efficiency of a taffy puller or any other mixing action. In particular, we focused on the mixing dynamics under pseudo-Anosov homeomorphisms, which we can study using train tracks. In the first section of this paper, we give the motivation for this problem and provide some background on surface homeomorphisms, mapping class groups, and the Nielsen-Thurston Classification Problem. We also give an example of how we can use train tracks to study surface homeomorphisms and find stretch factors of pseudo-Anosov homeomorphisms.

The second section of the writeup deals with an algorithm by Margalit, Strenner, and Yurtas in [4], which finds invariant train tracks in quadratic time. This algorithm uses split-fold sequences of train tracks. Our goal for this project was to give a full description of these split-fold sequences that can be used by a computer to run the algorithm. We explain our method for finding the split-fold sequences used in the algorithm.

1.1 Motivation

We begin with an explicit and famous example of a symmetry of a surface on a four-times punctured sphere, $S_{0,4}$. We present this symmetry using a real-life example: a taffy puller (see Figure 1). We can think of the knobs of the taffy puller as punctures on the sphere and the outline of the taffy as a simple closed curve lying on $S_{0,4}$. Suppose we want to measure the efficiency of the taffy puller, where efficiency refers to the rate at which the ingredients in the taffy are mixed. Notice that we are implicitly examining a symmetry of the four-times punctured sphere: the pulling action describes a symmetry. We are also studying the dynamics of this symmetry by measuring how efficiently our taffy puller moves taffy particles around.

More precisely, we notice that the pulling action first swaps the middle point with the right most point and then swaps the left most point with the center point (see Figure 2). We can iterate the pulling action on the taffy, our simple closed curve, to observe how quickly our taffy lengthens (see Figure 3). One way of measuring efficiency is to examine the cross section of the taffy and count the number of layers after each iteration. Notice that the pulling action is rather disruptive, and at any instance, there are no fixed points. In fact, we are particularly interested in studying the dynamics of such violent mixings, known as *pseudo-Anosov homeomorphisms*, which we introduce in a later section. We are also interested in measuring the extent to which our simple closed curves are stretched, or equivalently how efficiently various rigorous actions mix our surface around.

1.2 Train Tracks

We first introduce a way of defining surface symmetries: the mapping class group on a surface S, denoted $\mathrm{Mod}(S)$, the homotopy classes of orientation-preserving homeomorphisms on S that fix the boundary. We would like to examine the iteration of our mapping class group actions on curves. Unfortunately, curves

Figure 1: A taffy puller. [3]

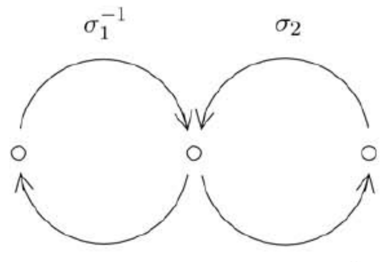

Figure 2: A pictorial diagram of taffy pulling. We can imagine our surface as a thrice-punctured disk. [2]

become increasingly difficult to visualize as more and more iterations are applied. For example, take the taffy pulling action on $S_{0,4}$ mentioned previously. If we draw line segments normal to the curve through the left side of the center and right punctures and count the number of intersections (see Figure 4), we notice that the numbers grow very fast.

To solve this issue, we can homotope parallel strands of the curve together (see Figure 5). One could preserve the complexity of the new strands based on how many strands were pinched together. Such homotoped curves are known as *train tracks*, and the homotopy is known as *zipping*.

We now present a more precise definition for a train track.

Definition. A *train track* is a weighted graph that satisfies the following conditions:

1. Every edge, called a *branch*, is smooth.

2. Each vertex, known as a *switch*, has a well-defined tangent line that defines two sides of the switch.

3. The weights, or labelings of the edges, satisfy the *switch condition*: the weights on the two sides of the tangent line must sum to the same value.

The switch condition allows us to represent the train track, denoted τ, on $S_{0,4}$ in Figure 5 as a vector in $v \in \mathbb{R}^2$, as two weights are sufficient to completely define τ. The mapping class group action, denoted f, is piecewise linear, so we calculate a 2×2 transition matrix M for f. To observe how the weights of τ increase over iterations, simply calculate $M(v)$. To calculate M, it suffices to observe the action of f on τ (see Figure 6).

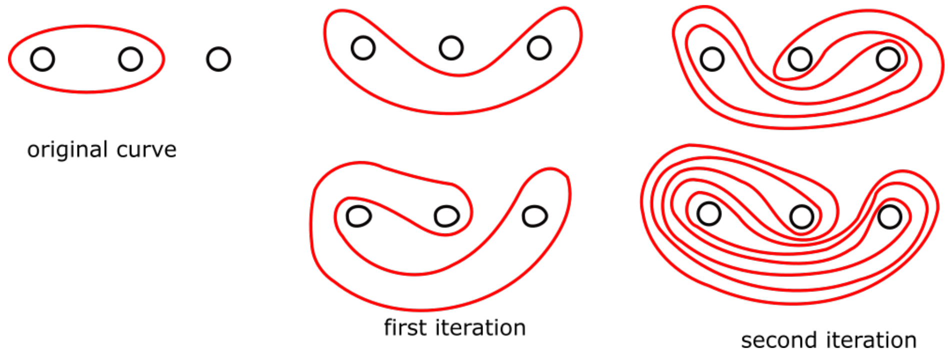

Figure 3: The red curve, a simple closed curve on $S_{0,4}$, represents the taffy. Measuring the efficiency of the taffy puller is the same as measuring how quickly layers of the taffy accumulate.

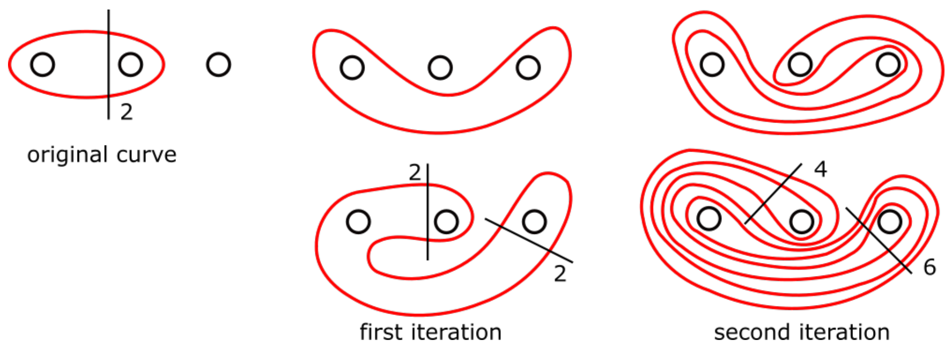

Figure 4: We count the number of intersections with the normal line segments to see how fast the pseudo-Anosov homeomorphism increases the complexity of the surface.

We take the basis vectors of $\mathbb{R}^2$: $b_1 = \begin{pmatrix}1\\0\end{pmatrix}$, and $b_2 = \begin{pmatrix}0\\1\end{pmatrix}$, which represent train tracks that loop around only two of the punctures. Let x and y be the weights on our two basis train tracks. From applying f on our basis train tracks, we see that $f(b_1) = \begin{pmatrix}2\\1\end{pmatrix}$ and $f(b_2) = \begin{pmatrix}1\\1\end{pmatrix}$. Then $M = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$.

The rest follows from linear algebra. We notice that M has two eigenvalues:

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{3-\sqrt{5}}{2}$$

Additionally, the corresponding eigenvectors are $v_{\lambda_1} = \begin{pmatrix}\frac{1+\sqrt{5}}{2}\\1\end{pmatrix}$ and $v_{\lambda_2} = \begin{pmatrix}\frac{1-\sqrt{5}}{2}\\1\end{pmatrix}$.

Suppose we want to find the nth iterate of M on any arbitrary train track τ, where $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ are the weights on the right side of the center switch, as seen in all the above diagrams. Then:

$$\begin{aligned} M^n(\tau) &= \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}^n \begin{pmatrix}x_0\\y_0\end{pmatrix} \\ &= \begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\1 & 1\end{pmatrix} \begin{pmatrix}\frac{3+\sqrt{5}}{2} & 0\\0 & \frac{3-\sqrt{5}}{2}\end{pmatrix}^n \begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\1 & 1\end{pmatrix}^{-1} \begin{pmatrix}x_0\\y_0\end{pmatrix} \end{aligned}$$

Notice that $\lambda_1 > 1$ and thus determines the limiting behavior of $M^n(\tau)$. This suggests that λ_1 is the

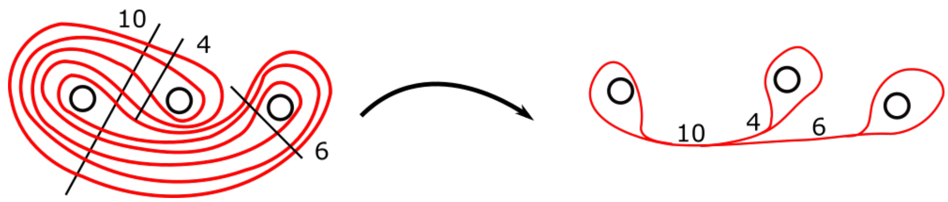

Figure 5: We can homotope a complicated curve into a train track. Notice that the two sides of the switch both sum to 10, satisfying the switch condition.

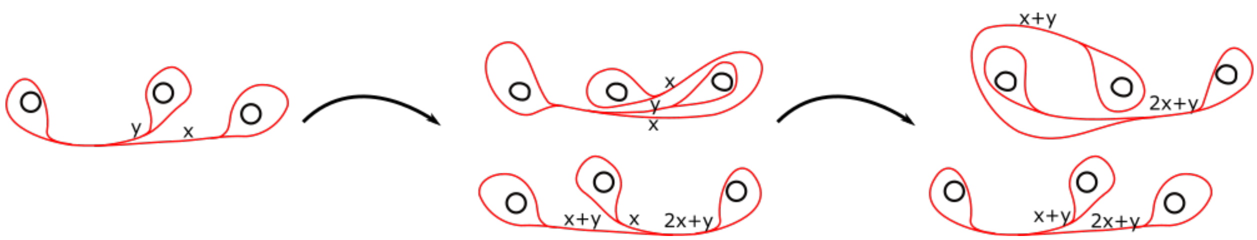

Figure 6: We can also apply f to τ.

rate of growth of f. In the next section, we will introduce the stretch factor, a precise way to measure the growth rate of f.

1.3 The Nielsen-Thurston Classification Problem

We first provide some background about how the stretch factor fits into the bigger picture. In the 1970s, the Fields medalist William Thurston classified all elements of Mod(S) into three types, known as the Nielsen Thurston Classification.

Theorem 1 (Nielsen Thurston Classification Theorem). *Let S be a finite-type surface. Then any element of Mod(S) is either:*

1. *periodic: finite-order,*
2. *reducible: fixes some finite union of essential simple closed curves on the surface,*
3. *pseudo-Anosov.*

The third type of homeomorphism, the one we are most interested in, includes the pulling action of the taffy puller mentioned in the previous section. Define a *foliation* of a surface S as a partition of S into "sheets" (see Figure 7). A homeomorphism is *pseudo-Anosov* if it stretches some foliation of S and compresses another foliation transverse to it (see Figure 8).

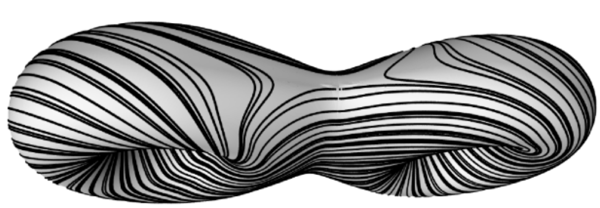

Figure 7: A foliation on a genus 2 surface [1].

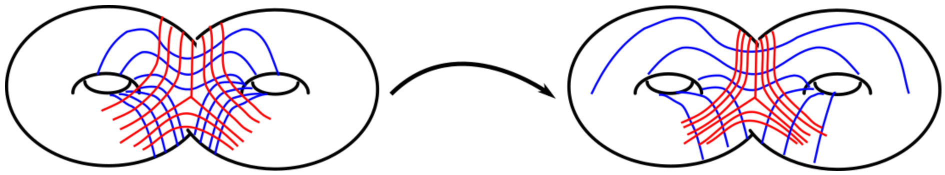

Figure 8: An illustration of an action of a pseudo-Anosov homeomorphism on a surface.The blue foliations are stretched and thus represent $\mathcal{F}_s$ while the red laminations are compressed and represent $\mathcal{F}_u$.

Definition. We call $f \in \mathrm{Mod}(S)$ *pseudo-Anosov* if there exist transverse measured foliations $(\mathcal{F}_s, \mu_s)$ and $(\mathcal{F}_u, \mu_u)$ such that f stretches $\mathcal{F}_s$ by λ_f and compresses $\mathcal{F}_u$ by $\frac{1}{\lambda_f}$. We call $\mathcal{F}_s$ the *stable foliation*, $\mathcal{F}_u$ the *unstable foliation* of S and λ_f the *stretch factor* of f.

Having defined the Nielsen-Thurston classification of $\mathrm{Mod}(S)$, we are now presented with a problem:

The Nielsen-Thurston Classification Problem. Given $h \in \mathrm{Mod}(S)$, classify h into the three Nielsen-Thurston types and if:

- periodic: find the order.
- reducible: find the fixed multicurve, known as the reducing curve.
- pseudo-Anosov: find the stretch factor.

In [4], Margalit, Strenner and Yurttas give a quadratic-time algorithm for the Nielsen-Thurston Classification Problem that finds stretch factors by finding invariant train tracks. This algorithms uses *split-fold sequences* of train tracks. The goal of the next section is to describe these sequences and how one can find them.

2 Finding invariant train tracks via split-fold sequences

2.1 What are split-fold sequences?

For a simple example, first consider a punctured surface, which has an ideal triangulation. Informally, standard train tracks are train tracks that are as simple as possible with respect to the triangulation. For example, see the train tracks on the left and right of Figure 9.

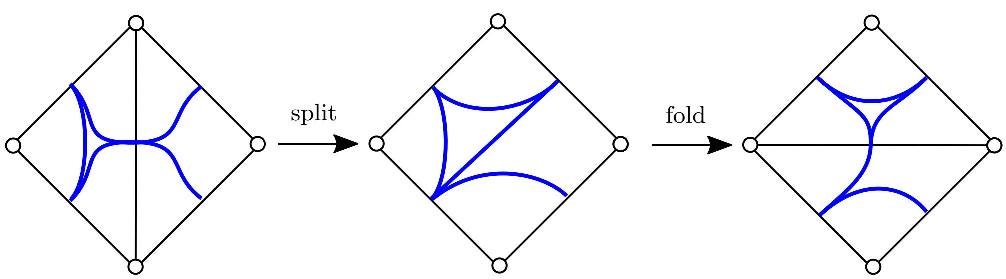

Figure 9: For punctured surfaces, the change of standard train tracks is straightforward to describe: it involves at most one split and one fold.

When the triangulation is changed, the standard train tracks also change. We would like to understand how they change when we apply a sequence of splittings and foldings to the old standard train tracks to reach the new ones. Figure 9 illustrates a split-fold sequences after a flip (the simplest possible way to change a triangulation).

Closed surfaces do not have ideal triangulations, so we need to use pants decompositions instead. In this case, standard train tracks (called *Dehn–Thurston train tracks*) are considerably more complicated to define, so the split-fold sequences are more complicated to find. Instead of flips, we now have two types of elementary moves, shown on Figure 10. Our project aimeed to find a description of all the splitting-folding sequences of Dehn–Thurston train tracks.

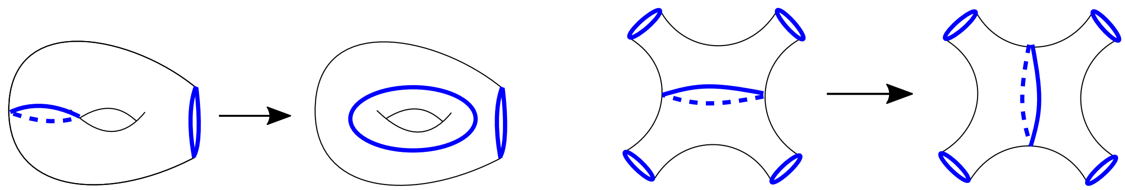

Figure 10: Elementary moves of pants decompositions.

Figure 11 shows an example of such a sequence. The illustrated case is one of the simplest ones—in the most complicated cases, about ten splittings and ten foldings need to be performed. In the next section, we will describe how to construct the split-fold sequences in general.

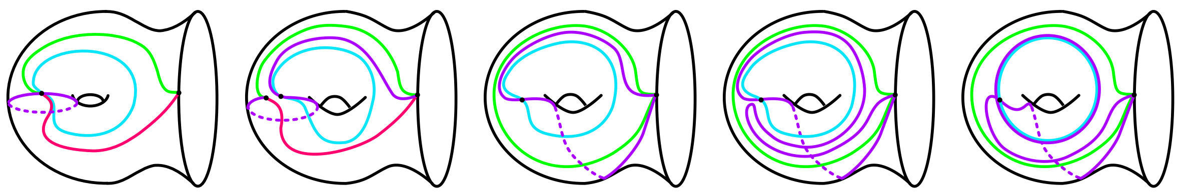

Figure 11: A splitting and folding sequence for Dehn–Thurston train tracks. The first step is a splitting. In the second step, the red branch is collapsed. In the third step, an isotopy is performed. In the last step, the purple branch is folded on the blue branch.

2.2 Finding split-fold sequences

Consider the figure below and the branches incident to the central switch. We say a branch is *left-leaning* if it travels towards the left half of the punctured sphere. For example, the dark pink, dark blue, and orange branches on Figure 12 are left-leaning. Similarly, every *right-leaning* branch travels in the direction of the right half of the sphere. Thus, the light pink, brown, and light blue branches on Figure 12 are right-leaning. Notice that although the light pink branch starts out on the left side of the sphere, it travels towards the right side. We also establish the convention that the purple branch is left-leaning on the left side of the central switch and right-leaning on the right side. Our goal is to apply a sequence of peelings until we reach the *stopping condition*: when all the branches incident to and on the same side of the central switch lean the same direction. Notice that the stopping condition is not fulfilled in Figure 12; in the top half of the sphere, there is one left-leaning branch and two right-leaning branches. Furthermore, on the bottom half of the sphere, there are two left-leaning branches and one right-leaning branch.

We now present a peeling sequence for reaching the stopping condition. Without loss of generality, we apply the peelings on the top pair of pants. If we are fortunate, we are able to peel all branches incident to the central switch off the purple pants curve (see Figure 13). In this special case, all branches will lean to the left on the top pair of pants. Thus, according to our stopping condition, there is no need to peel further on the top half. In order to reach the stopping condition on the bottom pair of pants, we need to peel the dark blue and orange branches off the pants curve in order for all the branches to lean to the right.

Sometimes we require a longer sequence of peelings. As a general rule of thumb, we peel branches with smaller weights off branches with larger weights until we can no longer do so.

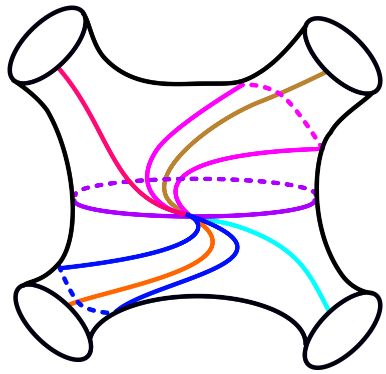

Figure 12: A local picture of a Dehn-Thurston train track.

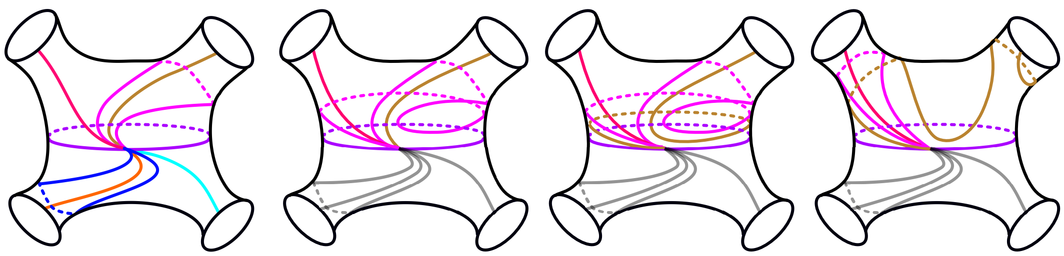

Figure 13: An easy example of a splitting sequence.

References

[1] Danny Calegari. *Danny Calegari's Home Page*. (Accessed on 07/07/2017). 2012.

[2] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups*. Princeton, New Jersey: Princeton University Press, 2011.

[3] Devin Lopez. *SugarPop's Candy & Soda: September 2014*. (Accessed on 07/07/2017).

[4] Dan Margalit, Balázs Strenner, and Öykü Yurttaş. "Fast Nielsen-Thurston classification". In: *In preparation* (2017).